THE MATHEMATICAL GAZETTE

EDITED BY
T. A. A. BROADBENT, M.A.

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Intending members are requested to communicate with one of the Secretaries (G. L. Parsons, Peckwater, Eastcote Road, Pinner, Middlesex; Miss Punnett, 17 Gower St., London, W.C.1.). The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette".

Change of Address should be notified to Miss Punnett. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

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ANDREW RUSSELL FORSYTH

Sc.D., LL.D., Math.D., F.R.S., Hon.F.R.S.E.

President, January 1903–January 1905
January 1936–January 1937

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VOL. XXI. FEBRUARY, 1937. No. 242

THE MATHEMATICAL ASSOCIATION.

THE Annual Meeting of the Mathematical Association was held at the Institute of Education on 4th and 5th January, 1937. On Monday, 4th January, the proceedings opened at 2.15 p.m. with the transaction of business; the President, Professor A. R. Forsyth, was in the chair.

The Report of the Council for 1936 was adopted.*

The Hon. Treasurer's statement of accounts for the year ending October, 1936, was adopted.

It was proposed from the chair, and carried, that Rule 5 be amended to read as follows:

> Anyone wishing to join the Association shall obtain from one of the Secretaries a form of application, which shall be completed and returned to a Secretary together with the first subscription. The form of application for Junior Membership shall contain a statement, to be signed by a lecturer or tutor of the Institution attended by the applicant, certifying that the latter satisfies one of the conditions mentioned in Rule 6 (b).

The new and amended regulations as to committees, made by the Council, were reported.

On the nomination of the Council, Professor L. N. G. Filon, F.R.S., was elected President for the year 1937. The existing Vice-Presidents, Treasurer, Secretaries, Librarian and Editor of the *Gazette* were re-elected. Professor Forsyth was reinstated as a Vice-President, having held that office up to the time of his election as President. Mr. F. C. Boon retired from the Council, and did not

* See pp. 3–6.

wish to stand for re-election. The following were elected as members of Council for 1937 :

Mr. C. T. Daltry,
Miss M. J. Griffith,
Professor G. B. Jeffery,
Mr. W. J. Langford,
Miss W. M. Lehfeldt,
Mr. C. W. Parkes,
Mr. A. W. Riley,
Miss G. K. Stanley,
Mrs. M. E. Williams.

Professor Forsyth then delivered his Presidential Address : *Applied Mathematics in School Teaching ; some General Considerations.** This was followed by Professor Georg Wolff's lecture : *The Development of the Teaching of Geometry in Germany.*†

On Tuesday, 5th January, with Mr. A. W. Siddons in the chair, the proceedings opened with a discussion : *The Teaching of Elementary Astronomy,*† introduced by Mr. J. A. Edgar, Mr. R. L. Marshall and Dr. L. E. Lefèvre ; this was followed by a paper on *Population Trends,*† by Dr. R. R. Kuczynski, and by a paper on *The History of Mathematics ; its Relation to Pupil and Teacher,*† by Mr. R. M. Gabriel.

The afternoon meeting was addressed by Dr. A. C. Aitken on *Arithmetical Recreations* † and by Mr. A. Romney Green on *Geometrical Design and some Practical Applications.*†

A Publishers' Exhibition was open during the two days.

GLEANINGS FAR AND NEAR.

1088. In the present state of science the most universal standard of length which we could assume would be the wave length in vacuum of a particular kind of light, emitted by some widely diffused substance such as sodium, which has well-defined lines in its spectrum. Such a standard would be independent of any changes in the dimensions of the Earth, and should be adopted by those who expect their writings to be more permanent than that body.—Clerk Maxwell, *Treatise on Electricity and Magnetism*, I. [Per Mr. C. Fox.]

1089. A Mediaeval Form of the Ruler and Two Pennies Experiment. Petit-Jean . . . was . . . the son of Henry de Cousin, master executioner of the High Court of Justice. The Constable Saint Paul was executed by him with such dexterity, that the head, when struck off, struck the ground at the same time with the body.—Sir Walter Scott, *Quentin Durward*, footnote to Chap. VI. [Per Mr. J. T. Combridge.]

1090. Three county cricketers made not out centuries on Saturday. Yesterday they went on batting and—each of them ended his innings with precisely the same total.

The odds against such a thing happening are literally infinite. It is just " one of the things that happen "—at cricket.—*Daily Express*, July 8, 1936. [Per Mr. S. F. Trustram.]

* See pp. 8–24. † To be published later.

REPORT OF THE COUNCIL FOR THE YEAR 1936.

DURING the year 1936, 168 new members have been admitted to the Association, of whom 48 are junior members. The number of members now on the roll is 1582, of whom 9 are honorary members, 104 are life members by composition, 3 are life members under the old rule, 1418 are ordinary members and 48 are junior members.

The Council regrets to have to report the deaths of the following members of the Association: Mr. T. M. A. Cooper, Dr. A. E. Cox, Emeritus Professor A. C. Dixon, Mr. J. H. Doughty, Mr. C. E. Haselfoot, Professor R. C. J. Howland, Mr. W. H. Macaulay, Emeritus Professor E. J. Nanson, Major W. Newbold, Mr. A. S. Percival, Mr. C. H. Sampson, Dr. W. F. Sheppard, Miss L. M. Swain, Mr. W. R. Thomas and Mr. W. R. Morgan.

Dr. Sheppard had been a member of the Association for nearly 30 years, had served for a great part of that time as a member of the Council and was President of the Association in 1928 and 1929. He took a great interest in the work of the London Branch and, until of late years his health prevented him, was a frequent attendant at its meetings.

Professor Howland will be greatly missed by the Southampton Branch, of which he had been a keen supporter since its foundation, in which he was largely instrumental, in 1931. He was its President from 1931 to 1933.

The Mathematical Gazette.

Increased space has been given to the interchange of opinion on specific teaching points. The number of reviews of textbooks in French and German is increasing, thus carrying on the policy of endeavouring to familiarise teachers in this country with foreign teaching methods.

The Branches.

The reports from the Branches make it clear that they are in a vigorous state. Particulars of the meetings have been given in *Gazette* insets. Membership has been well maintained. In some cases special efforts are evidently being made to attract into the Association a larger number of teachers from Elementary and Central Schools—a very desirable aim, for the number of members from such schools is at present unfortunately small. The Cardiff Branch has now increased its membership to the point which qualifies it for representation on the Council, and its representative has already attended Council meetings. The London Branch has, at the invitation of the responsible authorities, discussed the position of Mathematics as a group subject in the Higher School Certificate Examinations of the University of London. A memorandum drawn up by a special committee was adopted by the Branch and forwarded for consideration.

The composition of the Branches in terms of members, associates and junior members is as follows: Bristol, 13 members, 21 associates; Cardiff, 35, 41; Liverpool, 18, 57; London, 190, 90; Manchester and District, 36, 75: Midland, 38, 34, 1; North-Eastern, 39, 15, 1; Northern Ireland, 3, 30; North Wales, 5, 14; Southampton and District, 0, 20; South-West Wales, 12, 31; Yorkshire, 47, 76. The latest figures received for the over-seas Branches are as follows: Queensland, 11, 14; Sydney, New South Wales, 19, 122; Victoria, 7, 8.

The Library.

Members have used the Library much more during this year than during any previous year. A Fourth List, issued in February, extended the published Catalogue to the end of 1935, and the interest of its contents may have stimulated the requests for loans.

There have been many gifts, the largest being from the Royal Technical College, Glasgow, and from Mr. C. O. Tuckey. An exchange with the Centre National de Documentation Pédagogique, Paris, has enriched the Library by more than a hundred French textbooks, ranging from elementary arithmetic to analysis and higher geometry.

A bookplate, suitably worded, has been acquired for the volumes, several hundreds in number, which came to the Association from Mr. W. J. Greenstreet, either during his lifetime or after his death. Also a bookplate bearing only the name of the Association has been acquired for general use, and the work of inserting a label, completed in most cases with a donor's name, into every volume in the Library is proceeding systematically.*

The General Teaching Committee.

The General Teaching Committee has appointed a sub-committee to watch development and collect information about mathematical films. Consideration of the work of first-year specialists and of Junior Schools has been deferred.

The Boys' Schools Committee.

The Boys' Schools Committee has continued to work at the Supplementary Geometry Report, and members of the Girls' Schools Committee have cooperated with suggestions and criticism. The work has taken rather longer than was anticipated, but it is hoped to complete it by next Easter.

The Girls' Schools Committee.

The Girls' Schools Committee met twice during 1936. At the first meeting they discussed a draft of the Geometry Report, which was sent to them by the Boys' Schools Committee. At the second Mr. Gimson of Bedales very kindly came and read a most interesting paper on The Applications of Mathematics.

* See pp. 6-7.

The Problem Bureau.

The Secretary has been fortunate in obtaining the services of several new solvers, and he is very grateful for their valuable help.

Applications for solutions have been as numerous as before.

During the past year several questions, taken from public examinations, have proved to be incorrect. Some of the errors were due to inaccuracies in the numerical data; others were more serious, showing a complete misconception of fundamental principles. This state of affairs is by no means confined to one public examining body.

General.

During the year the Council has made new arrangements in connection with the work of the Secretaries and Treasurer. These have become necessary owing to the growth of the Association and the need for permanent headquarters. The Council has arranged on the one hand to rent a room in Gordon House in which the records, correspondence and other documents of the Association can be kept and clerical work done, and on the other to engage part-time clerical assistance—fortunately obtainable in Gordon House itself—to attend to some of the routine work of the Association.

It has been decided that, from 1936 onwards, the financial year of the Association shall end on October 31st. This change will make it possible for the Treasurer to present an audited statement of accounts at the Annual Meeting. No change has, however, been made in the date on which subscriptions are payable. They are still due on January 1st of each year.

The Council has drawn up a Regulation for a new Teaching Committee, the Regulation to come into effect in January 1938, when the term of office of the present Teaching Committees expires. A copy of the Regulation is being circulated with the notices of the Annual Meeting.

In view of the large increase in membership since the last issue of the List of Members, the Council has decided to print a new list in 1937.

The Council has arranged that the Association shall in 1937 be affiliated to the Conference of Educational Associations held annually at University College, Gower Street, W.C. 1. Members of the Association are consequently members of the Conference and may attend all meetings except those marked in the programme as for members of other Associations.

Officers and the Council.

The Council desires to record its appreciation of the services rendered to the Association by Professor A. R. Forsyth during his second term of office as President.

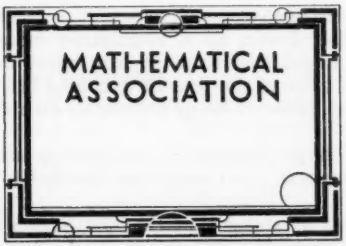
The Council nominates Professor L. N. G. Filon as President of the Association for the year 1937.

Since April Mr. F. Beames has been unable to reach the Library, and he sees little prospect of resuming in the near future the work voluntarily undertaken in 1926 and described in the Council's Report for that year as the least attractive part of the routine work of the Library. The Council, through the Librarian, is deeply indebted to him for ten years of zealous unobtrusive help. Mr. W. H. Jex, who has for some years past assisted the Honorary Treasurer, has now relinquished his appointment. The Council wishes to express its thanks to him for his services.

The Council extends its thanks also to Mr. T. A. A. Broadbent, Professor E. H. Neville and Mr. A. S. Gosset Tanner for their valuable and unremitting services to the Association as Editor, Librarian and Director of the Problem Bureau, respectively, and to Mr. F. C. Boon for his cooperation in the affairs of the Council since 1933.

THE LIBRARY: TWO BOOKPLATES.

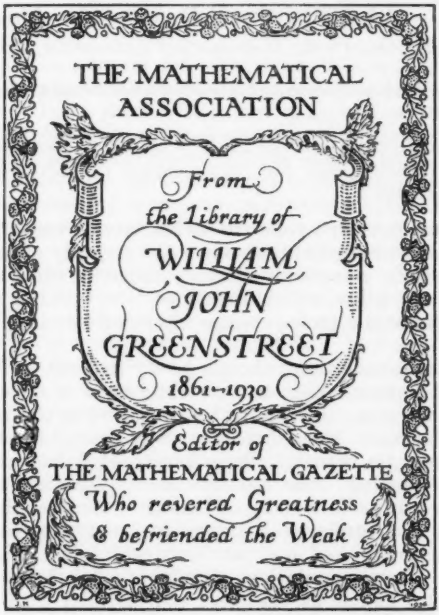
REFERENCE is made in the Report of the Council for 1936 to two bookplates acquired for the Library. For the private owner a bookplate is a luxury, but for the lending library it is a necessity if the risk of loss is not to be excessive. In the plate designed for general use, there is a space for the donor's name, and any member who



would like to see what his name looks like in our frame has the means in his own hands. Since in each case a record must be examined before a label can be completed for insertion, the labelling of 4000 volumes is a long task, but it is being performed systematically, and the occasion is being taken to check catalogue entries and cross-entries.

The Library contains several hundreds of volumes which the Association owes to the devotion of one man, Mr. Greenstreet. For many years there was hardly a month in which he did not give us old and valued treasures and unresisted bargains: "I thought the Library ought to have this". On his death in 1930 we came into

possession of all his mathematical books and periodicals, other than modern textbooks. It was evident at once that we ought to have a special plate for books that had been his, but it was only after



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discussions and delays that must seem excessive to anyone who did not take part in them that the final design was prepared. The plates have been printed and given to the Association by Mr. G. O. Tayler of Reading as a tribute to his old friend whose name they carry. The reproduction here should, to use the words of Mr. Pendlebury in a letter to the Librarian, " bring the memory of G. and his good deeds back to the minds of the forgetful and plant it in the minds of the ignorant ".

1091. Sir,—With reference to the headlines over the report of R100's flight to Canada in the *Newcastle Journal* this morning, may I point out that a speed of 10 knots is a speed of 10 nautical miles per hour and that, consequently, the expression " 10 knots per hour " means an acceleration (or retardation) of 10 nautical miles per hour, which has no meaning except to the mathematician, and certainly does not mean what I am sure your valuable paper intended it to mean. " Nautilus."—Letter to *Newcastle Journal*. [Per Mr. J. W. Stewart.]

1092. A force of high velocity had been unloosed on the world.—John Buchan, *Castle Gay*, ch. xx. [Per Mr. J. Buchanan.]

APPLIED MATHEMATICS IN SCHOOL TRAINING: SOME GENERAL CONSIDERATIONS.

By Professor A. R. Forsyth.

(*Presidential Address to the Mathematical Association, January* 1937.)

In the Rules of this Association, there is a declaration that its object shall be

> to effect improvements in the teaching of Elementary Mathematics and the applications of Elementary Mathematics.

Having regard to such a policy in action, having regard also to the professional duties of what probably is the majority of the members, I must appeal for a measure of your indulgence when my address, in retiring from office as your President, is devoted to some reflections concerning the present range of elementary Applied Mathematics.

The appeal is not based, though justly it could be based, on a ground that I myself have had no experience of school teaching. Yet, if not the rose, there have been periods in the past when I lived near the rose. More than fifty years ago, at the initiation of what then was University College, Liverpool, I had experience of elementary teaching; the students were preparing for external examinations of the University of London. The first charter of the College had not conferred a power to grant degrees; and the Victoria University had not yet been established. Some of the classes were of an unambitious standard, much below the dream of scholarships at one of the older universities: beginnings were bound to be modest. A little later, on my return to Cambridge as the junior member of the mathematical staff of my old college, Trinity, the ordinary round of duty assigned me a junior's share of elementary teaching. In particular, there were classes in the rudiments of algebra and the contents of the early books of Euclid, as required for Part II of the Previous Examination—less formally known as the Little Go. It is the undoubted fact that, a few months earlier, this same work had apparently been known by the students so far as to let them pass the entrance examination; it must have been ignored sedulously in the interval before residence began. (Let me say that such work is not now included in the undergraduate range at Trinity.) But, in retrospect, I am grateful for the disciplinary experience which thus was mine. From those students specially, I learnt something of the craft of teaching: learnt to note the weaknesses and to help the struggles of young men, who neither were (nor wished to be) mathematicians in the making. From them also I learnt the useful lesson that my duty was to give them the something they needed rather than the something which fascinated me: I had to learn a little of the art of

beguiling some of them into the consideration of matters, to which they were stolidly indifferent or about which their intellectual powers had suffered premature atrophy. So just because of these remembered experiences, I am not an entire stranger to the work, and to difficulties that beset the work, of teachers of elementary mathematics.

Nor is there an appeal for indulgence because, being usually ranked as a pure mathematician, I am choosing to deal with applied mathematics. Again there will be an apologetic return to earlier years. My own training as a student occurred in the days of the old Tripos, the single and undivided examination: there was no postponed Part, with a schedule of higher subjects for selective choice. In that Tripos, the amount of advanced pure mathematics was slight: nearly the whole of the pure mathematics constituted a range also useful in applied mathematics. The main range of the examination bore the generous Newtonian title of Natural Philosophy: in memory, applied mathematics ruled the day and not a little of the night. My first published original paper, an undergraduate venture, had dealt with the motion of a viscous fluid. It was only after my first degree that my systematic study of pure mathematics began; and I have often wondered how Cambridge had maintained an autocratic loyalty to its own type of Natural Philosophy. Even so, in my own instance, the association with Applied Mathematics did not cease—quite the reverse. On the Trinity staff, regular duty (discharged systematically through eleven years) required a willing me to lecture on Analytical Statics, of solids and of fluids: on Dynamics, alike of particles and of bodies, in two dimensions and in three dimensions: oddly as it now may seem in record, also on Thermodynamics. Finally, in the Imperial College of Science, parts of my courses of lectures related to the mathematics of electricity, strength of materials, simple vibrations; occasionally also they gave what purported to be applications of pure mathematics to some questions in electricity and in hydrodynamics. So I do not and need not fear the kind of gibe sometimes hurled at the shoemaker who will not stick to his last.

Rather, my appeal is due to the fact that, while I have chosen to deal here with applied mathematics so far as that subject does come or may continue to come within a school range, a long time has passed since I have had any actual experience of both the pleasures and the difficulties of mathematical teaching in any of its stages, elementary, intermediate, advanced.

I

For a little time during this year of office my wandering fancy played with a different theme. It would have been a natural sequel to an address "Old Tripos Days at Cambridge" which, given nearly two years ago to the Swansea branch of our Association, has been published in the *Gazette*. The notion was to present an account of the

modified Tripos, without dwelling on minor intermediate modifications, from 1882 to 1910, when the order of merit (which had been retained solely in connection with the less advanced Part of the examination) finally disappeared. During nearly the whole of that period, I had been a Cambridge teacher in various ranges of mathematics, a frequent adviser of students in practically all ranges (and not in pure mathematics alone), and a not infrequent examiner. A little reflection, however, made me first hesitate, and then ultimately abandon the notion. That chapter of history is worthy of a place in academic records; there are others who can write the chapter, also from observation and from experience, with what may be the added advantage of having received their training during an epoch when the traditional Cambridge practices in teaching were being gradually modified. As my fancy weighed the matter, it came to the conclusion that the Association (whose primary object, after all, is only remotely concerned with Tripos doings, past or present) had recently devoted a more than adequate amount of attention to that range of mathematical thought. And I seemed to remember also some quite reasonable hints, about the too-professorial corporate activity of the Association at the occasional cost of other interests more germane to its main object.

Simultaneously, other and more dominating considerations were presenting themselves with growing insistence. In the first place, you may remember that, in his presidential address two years ago, Professor Neville contrasted the latest set of Tripos papers with his own Tripos papers a quarter of a century earlier, and he compared the entrance scholarship papers of like dates: in particular, he was able to declare, with frank modesty, that his schoolboy knowledge at the earlier date would have been not inadequate to the scholarship demands at the later date. You also may remember that, at our Annual Meeting last year, we had an account of current school work for scholarships at entrance to a University: four selected speakers (two, from the school side: two, from the University side) expounded their views: of them, one alone dealt with applied mathematics. Only at one stage in the ensuing brief discussion was attention given to the school range of that branch of our science.

In the second place, during the course of the civil year which has just ended, academic authorities have been turning their attention to school certificates. Thus, in May last, the relevant Syndicate of the University of Cambridge announced that they were reconsidering the certificate, in view of

> the needs not only of University candidates but of pupils who leave school for training colleges, industry and commerce.

The Syndicate added that they were taking schoolmasters into counsel, an obviously wise precaution. The enquiry is one of far-reaching ambition: in its process, it might review the whole course and the main purposes of school education. Likewise, during the

past year, the Convocation of the University of London had a discussion of the relations between its own matriculation and the school examinations held by itself; the matter was referred to the Standing Committee of that body for consideration and report. Finally, there have been communications between the University of London and the London Branch of our Association concerning exemption from the Intermediate Examination of the University by means of the Higher School Certificate examination. In all the documents which I have seen, there can be found discussion and criticism of the pure mathematical range: the syllabus for applied mathematics seems to escape challenge except for criticism of occasional types of questions.

No man can do more than make a guess as to a date when reports and recommendations may appear or when decisions may be made. From a long experience, in Cambridge and in London, I know that the mills of the academic gods grind slowly and that sometimes they grind exceeding small. Still, we may assume that the matters are under consideration.

And then there intervened what undoubtedly is a personal reason, individual to myself. Taking account of the scholarship discussion of last year and remembering Professor Neville's avowal the year before, I made an opportunity of seeing some sets of the papers for school certificates and for scholarships, especially the then most recent papers set by the college groups in Cambridge. The scrutiny proved interesting to an old mathematical hand: firstly, in a comparison with papers some of which were set by himself in years now distant: secondly, in a comparison with the papers which were set to himself when he was a candidate in a year now very distant. It was a revelation to find an extraordinary divergence between the changefulness of a University advanced course and the almost sphinx-like changelessness of a school course, in so far as such examination papers provide trustworthy evidence, especially in applied mathematics.

II

Probably few mathematicians, young, or even of the present middle generation, have a general conception of the changes in English mathematical studies during the last sixty years. Let me choose Cambridge for more precise illustration. Mention already has been made of the preference which, when I was an undergraduate, was bestowed upon the applied mathematics of the day. After the division of the Tripos, which first took effect in 1882, there began an era of growing attention to pure mathematics; and through the intervening years, the detailed activities within that side of our science have changed as completely as have those of transport and other social service in the civil life of the community. Early in that time the theory of invariance, both algebraic and functional, held the field. Cayley's algebraic work still was a stimulus: Sylvester,

having discovered a new world of functional invariants, was firing Oxford disciples with his reciprocants : Lie's theory of continuous groups, with its varied developments, was beginning to be known. The alert devotion to much of that formal theory has now subsided, almost into a sleep and a forgetting. There succeeded another period when analysis was acclaimed—the analysis constituted by functions of a complex variable, alike in the general range of their theory, in the range of special classes of functions (particularly the theory of periodic functions of all kinds), and in applications of the general results to the whole range of partial differential equations : some of that bygone activity has become like the snows of yester year. At the present time, there are two ranges of preferred activity in Cambridge pure mathematics, each of acknowledged significance and each of recognised distinction in the world of mathematical science : one is a school of algebraic geometry ; the other may be described (briefly, though not completely) as devoted to the analytic theory of number.

Nor has the mode of awarding high distinction and recognition been less changed. In the old day, the sole medium was by examination in set questions ; and a successful student knew the beginnings of many subjects. In the new day, a dominant influence is research, leading to an early doctorate ; and a student who pursues that course can claim to be an expert in his specialised subject, itself restricted. To one who has lived in each of these periods, nearly all the old ways have now become new.

While the study and the range of pure mathematics have undergone such fundamental changes, an even more radical transformation has modified applied mathematics or mathematical physics. Within this domain, the change is due partly to the influence of Maxwell's work, partly to the scientific discoveries of the last forty or fifty years and the ensuing new world of half-speculative theories. Before these discoveries, the phrase "atomic physics" would hardly have carried significance : now, it is a commonplace. In those older days, what we used to call the "solution of the universe" was to be found in rational mechanics, with matter as the foundation : now, those old mechanical theories seem as dead as Caesar, with none so poor to do them reverence. The very constitution of matter itself absorbs the life of a whole army of physicists, whether experimental or mathematical. The field of high applied mathematics has changed as in an apocalyptic vision : we dream of a new heaven and a new earth—our old universe has passed away.

Thus half a century has seen vast changes in all the rarer heights of our science. There has been only slight change (sometimes all but invisible) on the lower slopes, constituted by the range of school work from its beginnings up to the requirements for certificates and scholarships. Of course, a corresponding amount or a corresponding quality of change could not be expected. There are, it is true, new presentations of old things. Progress has been greater

in school teaching, as evidenced by some of the examination papers for school certificates, than it has been in University demands, as evidenced by the set of entrance scholarship papers already mentioned. And I do not forget the Report of this Association on the teaching of mechanics in schools, a report to which I shall recur almost immediately. Further, account must ever be taken of the insistent limitations that are imposed by early needs.

To indicate the comparative lack of change, I shall imitate the example of Professor Neville, who cited his scholarship papers of nearly thirty years ago, and shall cite my own scholarship papers at the same college practically sixty years ago. There is some modification in the pure mathematics, of method rather than of matter. My impression is that my schoolboy knowledge of pure mathematics, as it was in 1877, could have received respectable commendation if submitted in the Trinity papers of December 1935—subject to reproof (if not worse) for lapses in the stiffened doctrines of convergence and continuity, and also for clumsiness in attempts at inequalities. When the papers in applied mathematics set in 1877 are compared with the later set, a strange family likeness has persisted —strange, that is, if we think of the vast changes that have been effected in the applications of mathematics, whether in the range of pure science, or in natural philosophy, or in the range of engineering in the routine of the practical world.

Certainly if the old limitation of subject and topics within what we style statics and dynamics continues to be imposed and to be accepted, the questions are bound to belong to that limited round ; and, in the present relations between teaching and examination, there must be corresponding limitations in the teaching. If any modification at all challenges notice, I am inclined to the opinion that the old minimum demand concerning the still old applications has become slighter. Some concise rule, summarising a principle into a formula, is now used at an early stage : once the rule is introduced, the application is submerged, and the ensuing calculations become an exercise in pure mathematics. Indeed, questions of such a type, so common throughout papers on statics and dynamics in examinations, employ hardly more than the vocabulary of rudimentary statics and elementary dynamics to propound demands for strictly geometrical or strictly analytical results. They are not applied mathematics, as the term appeals to me, any more than the ancient schoolmen's estimate of the number of angels who could balance on the point of a needle could be regarded as an application of the Theory of Numbers to Celestial Mechanics. Anyhow, the applied papers set at Trinity in December 1935 might have been set to the candidates at Easter 1877 : and again I am inclined to a self-laudatory estimate that my youthful knowledge of the applied mathematics of that distant date could have secured a measure of success if submitted at the later date. And having regard to modern applications of mathematics, such a result is hardly less than surprising.

III

I have read, with a real interest, the careful and valuable *Report on the Teaching of Mechanics in Schools*, prepared by a representative Committee and published in 1930 under the auspices of our Association: and I conceive that, at every turn, it can prove a trusty counsellor and a wise friend to teachers of Mechanics. But my immediate concern is of a different character. Doubtless the gradual accumulation of the body of its selected topics would have seemed appropriate in the middle of the last century. Whately, in his 1856 edition of Bacon's *Essays*, stated (as an obviously acceptable judgment) that a knowledge of the principles of geometrical reasoning and the elements of mechanics constitutes a part of a polite education. Machinery had then developed in amazing fashion; and there was an interest, perhaps even an enthusiasm, about the rational mechanics of the universe. Yet the world of practice was changing and scientific knowledge was developing: and a little later, in the 'sixties, there had arisen a demand for the introduction of experimental work into the domain of Natural Philosophy at Cambridge—a demand subsequently met in part by the establishment of the great school of physics. But there was a reluctance to actual movement in such a direction, not of course any hostility to the progress of the scientific work, but a hesitation concerning the introduction of any such work in the training of a Cambridge student. The proverbial line had to be drawn: it can be found * in a published passage that now reads strangely:

> It may be said that the [result of the experiment †] makes a stronger impression on the boy through the medium of his sight, that he believes it the more confidently. I say that this ought not to be the case. If he does not believe the statement of his tutor—probably a clergyman of mature knowledge, recognised ability and blameless character—his suspicion is irrational, and manifests a want of power of appreciating evidence, a want fatal to his success in that branch of science which he is supposed to be cultivating.

Much has happened in the long interval. The science of dynamical astronomy, in its rule at Cambridge, has had to share its lonely throne with newcomers: with electrics, in vast variety alike in practice and in theory: with atomic physics, in general: with the new chemistry: to mention some of the main subjects which, in varied fashions, use mathematics. The scientific world of applied mathematics has been transformed.

It is only reasonable to recognise that the best pupils at school can hardly be expected to learn more than the rudiments of some of these subjects which admit mathematical treatment. Further, it may happen (not in apprehensive fancy alone) that, if he does

* Todhunter, *Conflict of Studies*, 1873, p. 19.

† The falling of a sovereign and a feather *in vacuo*.

set out along the indicated path, he will be critically warned that he is forsaking mathematics and changing to physics. The charge can be faced serenely: there is neither guilt nor treachery.

In the course of a pleasant investigation during a part of last summer, devoted to booksellers and publishers and libraries, I made it my business to see not a few publications, of the educational textbook type, which contain applications of elementary mathematics to some subjects in mechanics and physics: particularly those subjects which are as real and as frequent in the ordinary workaday life of the world now, as were the classical mechanics in the heyday of their unchallenged domination. It is true that, quite often, these textbooks do not present their information with the almost eerie deftness which renders so many current mathematical textbooks the trusty guides through the dreaded valley of examinations. Still, for teaching purposes, there exists an ample variety of topics admitting of mathematical treatment, some without the differential and integral calculus, some with the use of only its rudiments. In electricity and magnetism there is mathematical work which does not demand the expert knowledge of the electrician, any more than our canonical statics and dynamics demand the expert knowledge of the mechanical engineer or the civil engineer. Laws of force, properties of couples and of various centres (not of gravity alone), laws of motion and resistance and friction, apply to vessels on or under the sea and to ships in the air, not less freely than to a set of fancy forces along the sides of a polygon: or to a system of pulleys: or to the ladder still resting on rough ground: or to an Atwood's machine which was resurrected a year ago: or to heavy smooth balls in a box or a bowl: or to particles falling down tubes which, sometimes rough and sometimes smooth, usually have a special form lending itself to geometric detail. To the results from the wider fields may more justly be awarded the comprehensive title of applied mathematics.

Of course we shall be told (I seem to have heard the phrases in many connections) that the old subject is peculiarly well suited for teaching and for examination: by long use and skilled practice, it has become precise, clear, definite, with well-recognised limitations that are observed: and all this recommendation arises through the combined and long-sustained efforts of teachers, and textbook writers, and examiners. All due praise to them for their labours! Yet it is the old familiar type of plea, coming honestly and honourably from those who love ancient ways; if one does not accept the plea as sympathetically and as completely as it is proffered, the hesitation is not due to any love of unrest. My own refusal to accept is based on several grounds. In the first place, much of what is done is no longer of significant worth as knowledge (and be it noted that we are dealing with applications, not with ideal notions or with abstractions): and some remembrances of pupils in the past have left me with doubts as to the general educational value of appreciable portions of the subject. But, there, each

teacher can draw directly from individual experience. In the second place, there are several branches of applied subjects which, as to their beginnings, do admit (and may require) treatment by elementary mathematics; some of these almost claim their entry into the domain of applied mathematics. In the third place, the stimulus towards the acquirement of new knowledge on the part of a young student will not become feebler—it may even be fortified—if and when he should find that the new work is part of the active life round him: such a discovery can make for added zest, even for a desire of guided knowledge, not always easy to stir, sometimes hard to maintain in continuance. Other reasons also weigh with me, some of them utilitarian in ultimate outlook; but I do not dwell on them here and now, as my purpose is to submit a question rather than to propound an answer. You may note that, in the reasons just specified, there has been a comparative omission of teacher and examiner, of textbooks and examination regulations: the concern of thought is centred upon the pupil. It is the pupil's education, in the widest sense of that omnibus word, which looms largest in my present thought. I want to help the habit of thinking and of learning, more than the acquisition of any aggregate of accurate knowledge, far more than a ready facility in examinations, however useful in quality these possessions can prove, both at school and in the full activities of life. Further, I want this habit not specially for a few choice spirits, carefully selected, and sedulously trained for the transient rewards of school and university success: but also for the youngsters of merely good average ability, and not least for the diligent duffers, so many of whom do not find such strength as may be theirs until a later stage in life.

IV

There is no obvious advantage to be derived from an enquiry into remote causes: there is nothing here for reciprocal half-recriminations. Teachers would be found to have been influenced by the kind of questions examiners could ask under specific ordinances; examiners would be found to have framed questions, which are fair within those ordinances and which accord with appropriate teaching textbooks: teachers and examiners and textbook writers would confessedly be restricted within the bounds of schedules: and schedule-framers are apt to be incarnations of caution when changes of widespread influence are under consideration. Circumstances have made for a continuance that has something of monotony and also something of inflexibility; and I submit that the position is not satisfactory, either as regards the spirit of the education or as regards the beginnings of knowledge, when these are brought into comparative relation with the altered temper of an altered age.

What, then, is to be done? Every thoughtful person can ponder his own or her own individual answer; but a fortuitous concourse

of individual answers may not combine into a reply that is helpful in policy. Some teachers will be content substantially with the present order, may even find it very good as a working system, though they would declare a readiness to accept changes when they have become absolutely necessary. Others, in differing degrees and from diverse reasons, will not be similarly content; yet they have the spirit of our country which, in practical matters, moves by reform (often minutely graduated reform) rather than by revolution. For them, decision comes to be the result of reasoned compromise, not an inevitable consequence of logical processes; and indeed (if the vocabulary of our own subject may be used for a moment) the problem is not quite determinate, the variables are distressingly numerous, and universal data are not formulated.

While I was musing, my thoughts were spirited into a dream; in a vision, a mathematical symposium was presented to my mind's eye. There was a gathering of disputatious friends, as in a dialogue of Plato's: it had met to discuss the place which should be taken by our science, particularly the applied part of it, in the intellectual cultural training of growing youth. From time to time, a speaker probably would appeal to general principles in order to establish some cherished dogma: but the central purpose of the gathering was restricted and special. To secure adequate discussion of the matter, the gathering had been constituted on a generously broad basis: and differing ideals, even opposed ideals, were represented. Thus the earliest to be noted was an examination-devotee: his governance had dominated activities of school and university, had dominated even entrance to the highest civil service of the land, through the second half of the last century: in the fulfilment of his impartial tests, he is satisfied that teacher and taught are rewarded for duty done and are slighted for neglect; that his process is the best of all quick methods for the display of normal ability; that trustworthy results are provided, beyond reproach and beyond challenge.* The second to be noted was an expert in the theory of education: his ideal insisted on a proper balance of culture which must be attainable by any selection of studies made from a conflicting and bewildering aggregate; he was ultra-dogmatic in his axiomatic demand for a liberal education, and was ready to be generous even about his own preferences when the precise significance of that demand had to be expounded in detail. Next, there was a philosophical expert (at least, that seemed his passing description, in my dream), who was the stern upholder of sound and logical consistency in all things, a ruthless adamantine critic of all deviation from his own standards of clear and earnest thinking: something of a visionary in our English world, which is not always ready to hate inconsistency and which so often muddles through illogical blunders: though critical rather than constructive in his own contributions, yet he is

* He might be surprised by a Report (by Hartog and Rhodes, *An Examination of Examinations*, Macmillan, 1935) of the International Institute Examinations Enquiry.

invaluable in exposing the gaps that can subsist between a dogmatic statement and an asserted inference. Then there were two rival representatives of science (I dare not call them scientific experts, for worldly malice sometimes assigns an ungenerous meaning to the name). The first of these was a fervent apostle of the study of his subject in all the ranges of its development, as one of the noblest occupations of the human intellect, worthy of absolute devotion on its own account alone : his ideal is research, practical or theoretical : and his quest is the discovery of budding genius, so that mankind may attain to ever-increasing knowledge. The second representative of science was less ethereal than his colleague ; certainly he was concerned with the increase of knowledge for its own sake, yet he could be critical of much that was labelled research : his preference was for a broad basis of science in several ranges, and he was wont to exercise a restraining influence on early or (as he termed it) premature specialisation, alike in his own field and in the fields of other workers, whether scientific or not.

Naturally the practical craft of active teaching was represented in this select gathering : no one person, alone, could pretend to be an adequate exponent of the varied experience of the great body of teachers in the country. Here also a couple of representatives had been summoned. There was one of somewhat older standing, with much experience which he was ready to cite ; he had become rather cautious out of that very experience : he was not in the least of the Lord Melbourne type (was it not ?) with his repressive question, " Why not let well alone ? " : on the contrary, he was willing to consider moderate changes, though occasionally sceptical as to their predicted advantages. His colleague was of younger standing, eager, enthusiastic, perhaps not quite disciplined out of a sort of joyful impetuousness : facing facts, new and old, as fully and as fairly as his more experienced colleague, but without that colleague's respect for tradition and for established practice : not imprudent, yet apt to be a little impatient with what he still found to be the interminable sluggishness of things. These two representatives could claim the attention which, after all, is no more than the meed due to their line in life : it is they who are to make changes work, efficiently if possible : and they are judges of what can be attained for the endless variety of material under their care.

And my dream pictured one other member of the symposium, apparently a complete stranger, entirely different from the mathematicians in character and outlook and manner. His standard was a boasted common sense ; in aggressive moments, he had been known to demand " delivery of the goods " ; he recked little of academic details, as regards which he had more than a touch of Gallio in his disposition : he has his own notions of what he wants in the ultimate issue, however attained if only it is attained ; and faced with scholastic schemes and recommendations, he has had no hesitation in asking, " What is the good of it all ? " Perhaps an odd colleague in that gathering : something of a Philistine, to use Matthew

Arnold's adaptation of Heine's word : not, however, unrepresentative of one phase of the spirit of this age : yet a not untypical parent who wants the most effective training for his children. He would not be unequal to a luminous exposition of his own knowledge of the " needs of industry and commerce ", though the substance of his exposition might not conform to the standards of any academic board considering school certificates.

There were some others also, less marked in their individuality. The aggregate was a mixed assortment of advisers : yet, believe me, not stranger than, in my own experience, some committees have appeared at their first meeting. It might have been an agreeable and a perhaps useful task to prepare an appreciative abstract of the views expounded at the symposium, the strength or any weakness of each argument, noting the optimisms or the forebodings which usually appear in opinions, the clash of claim and counter-claim. But my passing dream ended before the ghostly debate had begun : so there is no record ; such a debate can be held, free from the bias of record. May I suggest that some of you, proceeding beyond the range of last year's discussion in this place, should cast yourselves for the respective roles according to individual likings and thus make a practical attempt at such a symposium ? From your numbers, you could supply omitted representatives, according to your experience and your sense of preferences ; and even if some representative already suggested cannot find an actively sympathetic exponent, it is a clarifying discipline to try and state the other side justly. Much time is likely to pass before there can emerge what would stand somewhat in the position of the judgment of a final Court of Appeal.

V

But whether any or many take a share in the discussions, let me sound a preliminary warning which may not be superfluous, before them, during them, after them. Time after time, in Councils, in Syndicates, in Committees, I have noted that wide divergence between final estimates can occur, though these ostensibly are based upon the same presented facts. Occasionally, it may not be overrash to say " often " rather than " occasionally ", divergence will be due to differences of opinion concerning the fundamentals of early education. It is seldom that there is any explicit formulation of such fundamentals, though implicitly they can dominate considered opinions ; and some explicit formulation at the very beginning might clarify some issues, might help to reveal that final judgments can differ because they are influenced by discordant initial assumptions concerning the aims and the means of training in school instruction.

These considerations do not affect, solely or specially, a subject such as applied mathematics : they affect every branch of secondary education : linguistic, literary, historical, artistic, scientific. But it is not quite prudent to ignore them, merely because their influence

is general. At any rate, and with your acquiescence, I wish to deal, briefly, with this matter of fundamentals; for all the later stages of secondary education, whatever be the selected sets of subjects in our own science or in any other group, are affected by these fundamentals. I think that, to some extent, I do see their effect upon the outlook of the various representatives that flitted through my dream : and, indeed, there still seem to be drumming in my ears the sceptical mutterings of that grim commonsense man : what is the use of all your high mathematics, of your dead Latin and Greek ? What is the use of it all, as a preparation for active life in the stream of the world ? Some answer must be ready, for my own peace of mind if not for the dulcification of the imaginary questioner : so let me submit my confession. It is phrased as dogmatically as most confessions : to me, it appears a string of platitudes : to some of you, it may seem to contain strange doctrines and rank heresies, all reprehensible.

The education of youth is a matter of epoch, of race, of nation. Its system can change from generation to generation, adapting itself to shifting needs : it can be different for kindred nations of the same race ; and the respective systems for distinct races may even have little in common. Thus the ancient Greeks—a set of small city states, mutual rivals during the age of their greatest glory —had a system devised for the service of the state and for that service alone, military and civil. One branch was physical (they called it Gymnastic) : the other branch was cultural (they called it Music, in the comprehensive sense of their meaning of the word). Yet, in practice, the system varied from one state to another : thus the Athenian result and the Spartan result were in sharp contrast. Among our own people at the present time there are claims that the special training for a technical vocation, which may have no national duty, should itself be deemed a liberal education : an extreme claim, yet not more extreme than some of the practices of our ancient universities during much of the last century.

In face of the diversity of aims and of action, the task of formulating the necessary elements of youthful education is not simple : when achieved, the result may smack of individual preference rather than general agreement. The later stages are even more defiant of agreed formulation. There would, however, be good accord in postulating mere reading, mere writing, mere simple arithmetic, as primary elements in youthful training. Some one (I think it was Huxley) has said that limitation to such a range is comparable with the action of providing a child with a knife, a fork, a spoon, but no food. We may agree not to halt here : as soon as details are specified, divergences of advice begin to emerge : so let me go my own way. I hold the further education in early years to be the rudimentary drill, for later use in the humaner life of a human individual, in addition to his service as a component part of the social mechanism. The brain, as the organ of reception and decision, must be trained to see things as they are ; trained to think on its

own account (a habit not easily acquired, when opposed by the reluctance of modesty or the torpor of laziness); trained to acquire knowledge, not solely as an end in itself, not mainly for reproduction with the click of the ready answer to a question from the examination-machine, but definitely for constructive use. Certain practices, always needful and even necessary, can be acquired only gradually: the habit of finding where and how some wanted knowledge can be discovered instead of expecting it to be spoon-provided: above all, a faculty of expression, clear alike in spoken word and in written phrase. There is insistent need of accuracy in observation and in measurement, together with the beginnings of constructive reasonings, for estimate, for inference, for judgment. All these attainments (whatever their respective degrees) can and should be supplemented by a provision of cultural means for the awakening of ordered curiosity, in the doings of man and in affairs belonging to the current life of the world. Other interests may be added, according to some average bent of a group or to the preference of a teacher, always as opportunity may admit. Such a training can constitute the preliminary for any service in life: it does not profess, it does not even pretend, to be a training for an immediate vocation, technical or professional.

VI

A sketch of a mode of securing these contributory requirements may be outlined, very briefly save for one topic: it is unnecessary to add that, here, my concern is solely with things secular. Once a pupil has acquired rudimentary facility in reading, in writing, and in simple arithmetic, I would introduce drawing, and a little simple practice with ruler and compass, to be a continuation and a modification of the writing: and the practice could lead to the beginnings of practical geometry, merging sooner or later into an introduction to theoretical geometry, in attempt, if not in success. As material to be associated particularly with the reading, I would use the subjects of history, and of geography, initially with history: the history of our own land at first, and in a range which does not come to a dead end (as it used to end) with the battle of Waterloo or the Reform Bill: the geography of the world at large, because of the world-wide dominions in our Commonwealth. After a time, something of the history of other peoples in active relation with our own people would be added, not disdaining a fragment of (what may be called) commercial geography, of significant importance in an age of swift and ever swifter transit. Fairly soon, and concurrently in another direction, there would be a training to make the pupil use hand and eye and thought together: by a science subject, such as the elements of inorganic chemistry or of physics, according to convenient preference, at any rate, the pursuit of a subject which will stimulate a desire to find things out, for himself and not solely in obedience to a dogmatic formula. Throughout this stage there should continue some progress along the main road of arithmetic: after a while,

but not after too long a while, the beginnings of algebra would be tried, growing out of the arithmetic somewhat in the same spirit as the beginnings of geometry would grow out of the slightly systematic drawing.

The anticipated effect of these subjects, as regards intellectual training, could be expounded in fuller detail : this is not the occasion. But you may observe that, as yet, nothing has been said concerning one other topic, a training in expression. The consideration of that topic has been reserved deliberately to the last, in order to indicate, not too summarily, my own estimate of its essential importance in all stages of school life. To foster the faculty of expression, some language, studied as a language, will provide the necessary discipline. In my judgment, the obvious choice in any land is the spoken language of that land : not either of the two ancient classical languages, even with their rich treasures of the thought and the literature that come to be the possession of the happy band of Scholars : not a foreign language, even with the alluring promise of serious utility in some vague future : for us, our home English. It should be used for all linguistic drill, for analytical grammar, for all syntax : most of all, for the constructive use by the pupil himself in what we usually call composition. As compared with any other language, whether ancient classical or modern foreign, the home tongue has the overwhelming advantage of a vocabulary, already familiar and growing without dictionary efforts, already in dynamic action under the influence of its organic laws, in current speech, and in a noble literature. There is no desire to proceed as far as the formal English section, which is set out for your school certificates: no more is demanded than a respectable knowledge of the language and a respectable use of the language, not at all the same as a stage in a scholar's philological attainments in English. Perhaps the regular writing of English, as a discipline in itself, is now more frequent than used to be the custom in an older day ; but I cannot forget some of my experiences as an examiner at Cambridge, for the Tripos in its different Parts, for scholarships at my own College, even (let it be said with bated breath) even for Fellowships. Complaints of careless quality in the writing of our language are a commonplace with some critics : we mathematicians are frequently charged with the offence, and a plea of not guilty cannot always be established ; but old observation compels me to think that, in the pillory of the critics, we shall have many fellow-culprits who are certainly innocent of mathematical proclivities. Be that as it may, my own opinion is that English should be the linguistic medium for linguistic training in the earliest stages of English education : and, further, my opinion would welcome some standard of attainment for all formal academic qualification.

I would ask you to note that all these later suggestions are concerned with a preliminary stage : their extent is less ambitious than the range of requirement for a lower certificate in the schools ; but there is a rather different balance in design. Neither in thought

nor in word is there, on my part, any wish to stir upheaval in school courses. I have merely submitted what seem to me the desirable constituents in a sound education for all who proceed beyond the knife-fork-spoon stage. The nature of that preliminary education affects, directly or implicitly, all estimates of the courses to be included in any detailed scheme for its fuller complement. Everything beyond these preliminaries belongs to the secondary range where, owing to the conflicting multiplicity of subjects that now have been pressed into the domain of school teaching, there is bound to be the beginning of selection and exclusion, even of specialisation, call it what we may. Were the power mine, I should defer all strict specialisation until the last stage at school: let me confess that I would like even to exclude it completely, an impossible aim in present circumstances. There are many young people (I am not thinking of the exceptionally gifted pupil or of the exceptionally slow pupil) to whom linguistic training will appeal in vain: gradually in our academic centres there has been an easing of the old demand which compelled at least a bowing acquaintance with the languages politely called learned. Others there are, not a few, to whom the aridities of mathematics, the barrenness of precise measurement, even the details of observational science, prove a wearisome quest. Too often there is the inevitable struggle between the demand for more discipline where it reasonably can be exacted, and the desire for more freedom where it profitably can be indulged: the coward in me feels relieved when the decision does not rest with me.

One conclusion, however, is obvious as regards the conflict of school studies. The subject of applied mathematics, whether imposed as a qualifying discipline, or selected as a free choice in a combined course, can be studied by only a small proportion of pupils at any school which is not technical in the main. Even the assumption that it belongs fitly to a school course, in the sense that it ought to be initiated as a school preliminary to a career at a university or at any institution of advanced study, might not secure acceptance either within my imagined symposium, or elsewhere generally. I cannot tell: I respectfully leave the whole matter to your consideration. But I would plead that, whether at school or beyond school, the beginnings of the subject should eschew the veiled opportunities for manipulative exercises in subjects which belong to pure mathematics, and that these beginnings should be connected, in fact and in spirit, with the measurable phenomena of the universe.

A year ago you did me the honour of electing me into the annual office of President. Many years earlier, while still actively engaged in teaching mathematics, I had already held the presidency: so the honour was signal indeed. I was grateful for the renewed distinction: at our last Annual Meeting, no opportunity of expressing my

thanks arose : let me thank you now. I thought then and think still, yet not ungratefully, that you might have been wiser to choose some President who was less senior in standing and had remained in closer association with your current work ; and I am glad to know that my successor is in the full course of his high mathematical and administrative career. During the past year, I cannot pretend (even silently) to have rendered any useful service. Fortunately for your general well-being, you can rely on your Council, on your Committees, and especially on your more permanent officers, as vigilant and active guardians of your interests. To them you owe, and in thought you may pay, a debt of gratitude for their labours—a declaration that, I trust, is permissible to a President now retiring from duty. For myself, the year has been happy, not least in the renewal of old experiences. In laying down my office, I would tender my wishes for the future of the Association in words from an ancient tongue : *Floreat, vigeat, crescat.*

1093. The most exquisite demonstration in mathematics, or the most pleasing disquisition in metaphysics, if it does not ultimately tend to increase some sensual satisfaction, is delightful only to fools, or to men who have by long habit contracted a false idea of pleasure ; . . . —O. Goldsmith, *The Citizen of the World.* [Per Mr. S. F. Trustram.]

1094. In the neighbourhood of Oldham there are weavers, common hand-loom weavers, who throw the shuttle with unceasing sound, though Newton's *Principia* lies open on the loom, to be snatched at in work hours, but revelled over in meal times, or at night. Mathematical problems are received with interest, and studied with absorbing attention by many a broad-spoken, common-looking factory-hand.—Mrs. Gaskell, *Mary Barton*, Ch. 5. Mrs. Gaskell resided in Manchester from 1832 until her death in 1865. [Per Mr. J. B. Bretherton.]

1095. ARITHMETIC AMONG THE STARS.

She . . . can work all sorts of mathematical problems up to 1,000.—" Shirley Temple—the first seven years ", *Screen Pictorial Summer Annual*, 1936.

1096. SPACE AND TIME.

" It is difficult to withstand the suspicion that the three dimensions of space and the fourth dimension of time may be four independent variables of a system that is neither space nor time, but something else wholly unconceived by us. Our present enigma as to how a First Cause could itself have been brought into existence—how the tortoise of the fable, that bears the elephant that bears the world, is itself supported—may be wholly due to our necessary mistranslation of the four or more variables of the universe, limited by inherent conditions, into the three unlimited variables of space and the one of time."—Francis Galton, *Inquiries into Human Faculty*, 1890, p. 196. [Per Mr. H. Pfannmuller.]

1097. Grimmett and O'Reilly seemed to mesmerise the batsmen, except Hendren and Hart, and I have a strong suspicion that the latter's bat was not as straight as Euclid's famous definition when O'Reilly bowled him.—P. F. Warner, *Daily Telegraph*, May 29, 1934. [Per Mr. C. Blow.]

INDIRECT PROOF IN GEOMETRY.*

A Defence.

By J. G. Leadbeater.

One frequently hears mentioned as a point of superiority of the New Geometry over the Old that it employs fewer indirect proofs, the implication being sometimes that these are less easily followed than the direct, and sometimes that they are less convincing. There is, in fact, no argument more readily understood or more convincing than an indirect proof properly so called. Its only inherent difficulty is that the figure necessarily seems to illustrate the possibility of that which according to the theorem is impossible.

To what then are we to attribute the disrepute into which the indirect method has fallen? In the writer's opinion it is due solely to bewildering and illogical presentations often given by writers of textbooks. To banish once and for all what we have called the only inherent difficulty, they have merely to impress upon the student that there is but a *seeming illustration*, and that it comes about by a distortion of the figure, *permissible* because it is made necessary by assuming the opposite of what is to be proved. They do not do this. Indeed, it almost seems that they have conspired to be dumb on the subject of distortion, and it is interesting to note that their chief sin of commission is that they themselves indulge in a distortion which is *not permissible*. To the utter confusion of the student, they often deliberately make a figure that does not embody the data, and yet assume in their later argument that this figure does faithfully set them up. We may possibly achieve logical proof by assuming the opposite of what we wish to prove; but argument from a figure made not according to specifications must somewhere involve the logical fallacy of an ambiguous term.

Another fault frequently noticed is the labouring of the proof with such words as *unless, etc.*, or *nor can any other, etc.* A fireman playing billiards in his upstairs quarters does not on hearing an alarm go down the stairs. He descends at once to the ground-floor by slipping down a smooth pole. The words *which is impossible* correspond to the alarm, and the student should at once descend to where he added to his data the assumption that the theorem to be proved was untrue. In every case his argument should now conclude with two formal steps. The first is to deny the added assumption. The second is to affirm hence that which was to be proved.

Consider the theorem that if two circles touch externally the straight line joining their centres will pass through the point of contact. It is now generally proved by using the common tangent at the point of contact; but it used to be proved indirectly. In Messrs. Hall and Stevens' edition of Euclid one finds a professed *reductio ad absurdum* in which the line of centres is drawn straight, but misses the point of contact because the centres have been taken

* A paper read to the Queensland Branch of the Mathematical Association.

low. This last is a modification of the data, and any property claimed to be thus established can be claimed only in respect of circles suffering from *lapsus centri*, for to say *Let A and B be centres* and to mark them where we *know* the centres not to be, is equivalent to enunciating in terms of circles with centres displaced. The claim cannot be made even in respect of all such circles, for one or both circles may have suffered a *lapsus* straight towards or away from the point of contact, or an upward movement of one centre may have been balanced by a downward movement of the other. In both cases the construction adopted fails, for the join of the displaced centres will fail to avoid the point of contact.

Suppose that we are asked if we can prove the theorem. We say *Yes.* Our friend says *Good. I'll draw the circles. There you are.* He has marked true centres A and B and the point of contact C. We say, *Well, we'll take the centres down here.* What will he say? His choice of words will depend on how well he knows us. He has every right to say, "You must be prepared to take a figure ready-made and tested for accuracy, and your own figure is always liable to a test. You have no more right to assume that centres displaced can retain the attributes of true centres than you have to assume that the circles may move apart and still be regarded as touching, or that circles may be drawn re-entrant and be regarded as the true circles required, or that six squared is forty-nine, though when convenient it may be taken as thirty-six, or than you have to say, in order to establish some property of a quadrilateral, *Let us assume that the quadrilateral has five sides.* You will get nowhere logically by ignoring facts already proved. One of these facts is that a circle has only one centre. You invite us to assume that it has two. We who know that it has not two refuse to make the assumption except for the express purpose of proving by a *reductio ad absurdum* that it has only one".

It is not necessary to displace the centres. Join AB and if the join does not pass through C, let it pass elsewhere and cut the circles at M and N. (We show this by making the join curve away from C.) Then ABC is a triangle on the base $AMNB$, and the base is clearly greater than the sum of the other two sides, which is impossible. Therefore the line of centres does not pass elsewhere than through C. Therefore it passes through C. Q.E.D.

If we are told that we too have pretended that something is what it is not, that a curved line is straight, the reply is that we have made no pretence but have assumed for argument's sake the opposite of what is to be proved, and that at least we have not monkeyed with the data. The perfect circles, their touching, and their true centres are facts unchangeable. What the line of centres will do is a matter of argument. Not mere ocular demonstration but theoretical proof is required that it will pass through the point of contact. Our proof consists in showing that the consequences of its not doing so are absurd. For the purposes of this argument its not passing through the point of contact becomes fact. Whatever

may happen to it in consequence, it is still regarded as straight, and ACB is by assumption crooked.

A champion of the proof first indicated may say, " May I not then for argument's sake assume that the centres have slipped down ? " Certainly he may if it amuses him ; but when he has finished he will still have to enquire into what happens when they do not slip down, for *lapsus centri* implies an abnormal condition, and the present discussion has to do with circles presumed to be healthy.

But—he pleads—our proof merely says that we are to suppose the line of centres to pass otherwise *if possible*. Just so. The line of centres, however, means *not some line of non-centres*, but THE *line joining centres that are rightly placed and stay put*. The *if possible* clause does not save the situation. It simply is *not* possible for centres to slip and still be centres, and omission to ask leave to displace them does not alter the fact that they have been displaced.

In the previous theorem where the circles touch internally the authors displace one centre.

Let us consider now p. 163 of their *Geometry* (1903) where they deal with the Converse : " If a pair of opposite angles of a quadrilateral are supplementary its vertices are concyclic ". They deliberately and quite unnecessarily make a quadrilateral not in accord with the data. They give no warning of this distortion, nor do they make it sufficiently gross to be obvious. Having reached an absurdity, they do not make at once the important inference to which, assuming their reasoning to this point correct, they are entitled. They do not give the formal close. Finally they say *that is*, a substitute for *therefore* which in Geometry should be discouraged.

They say, " Let $ABCD$ be a quadrilateral in which the opposite angles B and D are supplementary ". Is the student to interpret *let* and *are* to mean *We have made* ? Is he to believe that angles B and D of the figure are really supplementary ? The authors would say, " No ! He is merely invited to assume for argument's sake that they are ". In the writer's opinion the student naturally does believe them to be true supplements. The circle $ABCE$ having been drawn, he is expected to realize if necessary that he has been misled, that angle D was deliberately made not supplementary to angle B, and that for these angles to be supplements the point D should have lain elsewhere. Perhaps, since D is not far from E, he doubts that the circle has been truly drawn and suspects that if so drawn it *would* pass through D. He is encouraged to these thoughts by the words some lines down, *unless E coincides with D*. Though it is really D that is misplaced, the words last quoted imply that D is all that it should be and that the birth-stain is on E. They are utterly mischievous. They may, and should, be omitted.

Mr. H. G. Forder, on p. 161 of his text (1930), gives three figures for the converse under discussion with a *reductio ad absurdum* for each. Like Messrs. Hall and Stevens he is careful to draw an angle D not supplementary to B, notwithstanding his having said, " Given

ABCD is a quadrilateral, the angles *B* and *D* are supplementary ". Note the *is* and *are*.

His proof, though he gives three figures, is no better logically than that of Messrs. Hall and Stevens, for in every figure it involves, as theirs does, an angle which was made not what the data required it to be.

The outstanding feature of the figures of the authors mentioned, the exclusion of *D* from the circumference of a true circle through *A*, *B* and *C*, would be intelligible if they had set out to prove, not the converse attempted, but what may be called the *companion* to the proposition, namely: The opposite angles of a non-cyclic quadrilateral are not supplementary.

This may be readily proved by an analysis of the positions off the circumference of circle *ABC* which *D* may take. The converse then follows at once, for the theorem and its companion respectively give the logical forms I, All *A* is *B*; II, All not-*A* is not-*B*, from which the classes *A* and *B* are identical. In other words, All *A* is all *B*, a form of proposition which Sir William Hamilton proposed to denote by the remaining vowel *U*.

It was suggested in a short paper read at the meeting in Brisbane about 1907 of the Association for the Advancement of Science, that the method indicated should be used for the proof of all converse theorems. In the words of the press report the contributor, Mr. H. Tomkys of Charters Towers, chiefly claimed for it that it frequently obviated the necessity for the use of the *reductio ad absurdum* method, which had the objections (*a*) that it assumed that to be true which was obviously untrue, and (*b*) that it required that to be constructed which was experimentally impossible.

Falsification of the data is responsible for these objections, and neither applies to a *reductio ad absurdum* properly so called. Though the suggestion is valuable, the writer prefers to retain indirect proof even for converses, discarding only such examples as involve the *initium absurdum* of falsifying the data, and such as can be replaced by shorter direct proofs.

The keeping of *D* and *E* apart should be effected by falsifying not the quadrilateral but the arc *CA* of the circle, *E* being shown on the falsified arc, and *D* lying off this damaged arc but on a dotted repair of the circle. The figure thus exhibits the data faithfully set up, the opposite angles truly supplementary, and also that which was to be proved, *D* lying on the genuine circle through *ABC*. Moreover, the falsification of the arc is so obvious as to need no special mention.

Let *ABCD* be a quadrilateral having its opposite angles *B* and *D* supplementary.

Its vertices shall be concyclic.

PROOF. If the circle through *A*, *B* and *C* does not pass through *D*, let it pass elsewhere.

Then it must cut *AD* or touch it.

It cannot cut AD first at A, for it must then also cut DA produced in a point which would lie outside the circle.

A circle which touches a straight line cannot meet it in another point, and since circle ABC must pass through A in AD, it must, if it touches AD at all, do so at A.

Suppose the circle ABC to cut AD internally as in Fig. I, or AD produced as in Fig. II, or to touch AD as in Fig. III.

Join BD and let BD or BD produced cut circle ABC at E.

Complete the cyclic quadrilateral $ABCE$.

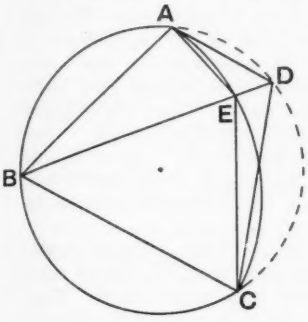


FIG. I.

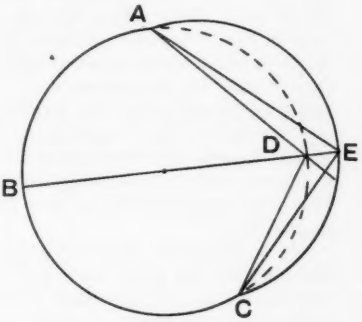


FIG. II.

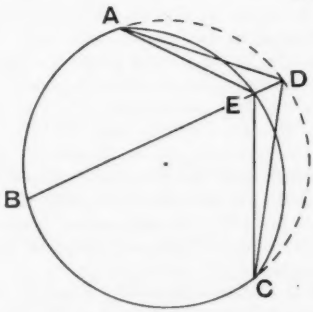


FIG. III.

In all three figures angles ADC, AEC are unequal by Euclid, I, 16, and as they are both supplementary to angle ABC they are also equal, which is impossible.

Therefore circle ABC does not pass elsewhere than through D.

Therefore it passes through D. Q.E.D.

OTHERWISE. (*The reader must draw his own figures.*)

Let the circle ABC cut AD or AD produced at E, or touch AD at A cutting CD internally at E.

Complete the cyclic quadrilateral $ABCE$.

The *reductio ad absurdum* is obvious, and one wording will suffice for all three figures ; but in connection with the third case, strictly, we should justify our assumptions that the touching will occur at A and that the cutting of CD will be internal. This adds somewhat to the length.

If the falsified arcs offend the eye, let it be remembered that we set out, not to draw pretty circles, but to prove a theorem. The circles are *ours* to distort where our assumption of the untruth of the theorem makes distortion necessary. Note that we have not distorted *your* quadrilateral. We have drawn it as we presume you would have done. The opposite angles *are* supplements. We are able to say, " but by *construction* angle ADC is the supplement of angle ABC " where Messrs. Hall and Stevens dare not say more than " but by *hypothesis* . . . ", an hypothesis which they deliberately negatived in the making of the figure. Since their figure is essential to their argument, we are justified in asking, " Are angles B and ADC supplements or are they not ? " It is useless to answer that they are *supplements by hypothesis, though not in fact.* If one of the premises is thus qualified, the conclusion cannot be unqualified.

On p. 161 they deal with an earlier converse : Equal angles standing on the same base and on the same side of it have their vertices on an arc of a circle of which the base is the chord. They make the given angles unequal, as does Mr. Forder in a three-figured treatment. The objections already urged apply here, and the remedy indicated is again available. The following list, to which additions could doubtless be made, will show that the authors already mentioned are by no means alone in regarding a tinkering with the data as permissible in an indirect proof.

Calling the converse last mentioned (I), and that of the cyclic quadrilateral (II), Messrs. Godfrey and Siddons (1904), and Mr. A. E. Ikin, falsify the given angles in (I) and the given quadrilateral in (II). Messrs. Baker and Bourne (1905) leave (I) as an exercise, suggesting a *reductio ad absurdum* treatment. They falsify the given quadrilateral in (II), though more courageously than do Messrs. Hall and Stevens. Messrs. Cracknell and Perrott (1925) falsify the given angles in (I) and by it prove (II). Messrs. Colville and Andersen (1934) follow closely the lines of Mr. Forder. They adopt his treatment of (I) and by it prove (II), an alternative which he mentions.

The practice of interfering with the data is evidently firmly established. It is nevertheless a futile proceeding. To reach from a false setting-up of the data, a figure giving a perfect illustration of the possibility of that which according to the theorem is impossible, and no illustration whatever of that which it is to establish, is, to say the least, a very unpromising opening for a proof of any kind. The student is persuaded that it all proceeds from the first step in an indirect proof, an assumption of the untruth of the theorem proposed. Nothing of the kind.

Express the converse which we have called (II) by: If C is D, A is B. To assume its untruth is to suppose that when C is D, A is *not* B. The authors mentioned suppose instead that when C is *not* D, A is not B, and the proof that they do so is that they build on this supposition the figure from which they propose to argue. Now, there is nothing absurd in the supposition they have made. Indeed, the theorem already proved, If A is B, C is D, and its companion (which can be proved), together justify not the mere assumption, but the assertion that if C is not D, A is not B.

How can a *reductio ad absurdum* be argued from a figure which illustrates no absurdity? This is how they persuade themselves that it can. They profess to build their figure on the supposition that when C *is* D, A is not B, some apparently considering that they absolve themselves for building quite otherwise, when they introduce the figure-producing clause with the pious preface, *If possible, let* . . . They reach an absurdity by insisting that the figure illustrates that C *is* D, adding by way of smokescreen the words *by hypothesis*.

OTHERWISE:
They have proved that All K is T.
They wish to prove that All T is K.

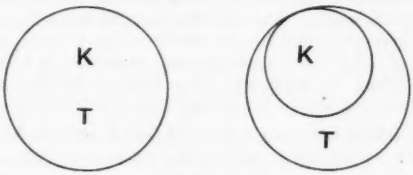


FIG. IV.

Either of the above figures is consistent with All K is T.

(If we prove, as we may, that all not-K is not-T, the circles T and K will coincide in Fig. II as in Fig. I. It will follow that all T is K. Also that all not-T is not-K. But the figures themselves suffice for our argument.)

From the figures it is clear that though there may prove to be absurdity in the supposition that a case of T lies outside circle K, *there can be no absurdity* in the supposition that a case of not-T lies outside circle K. Hence again the figures of the authors mentioned illustrate no absurdity at all, and the claim to have performed a *reductio ad absurdum* must be disallowed. A point P representing an instance of T must lie within the T circle. They ask for P within this circle for their enunciation, and, IF POSSIBLE, without for their construction, and, BY HYPOTHESIS, again within for their proof. We must refuse their request. There is no subtle virtue in the phrases *if possible* and *by hypothesis* which justifies this infraction of the LAWS of THOUGHT, for which see any textbook on LOGIC.

Note.—The above is portion of a paper read in the writer's absence before the Queensland Branch of the Association. It was thought by some of those present that it would be difficult to draw the figures here proposed for the quadrilateral. It is not so. Draw a circle with portion of the circumference merely dotted. Take B in the plain arc AC, and D in the dotted arc CA. Now (or earlier) join CA with the false arc which, seeing that the proposition is a theorem, may be drawn free-hand.

One member would not distort the figures at all. This plan was adopted by Pirani (1887). He does not falsify his data. Though in (I) and (II) he avoids a formal *reductio ad absurdum* his proofs for these are really indirect, for ($ABCD$ being in cyclic order) he says in (I), where AB is the common base, *Let circle ABC cut BD or BD produced in E. Join AE*; and in (II), *Let circle ABC cut CD or CD produced in E. Join AE.* His figure for each theorem shows D and E coinciding. Assuming them separate he proves by Euclid, I, 26, that they must coincide. In (I) his construction is such that a single figure suffices. In (II) he leaves undiscussed the possibility that circle ABC may touch CD at C. J. G. L.

1098. The Face in Numbers.

The Chamberlain celebrations are surely incomplete (suggests a scientific correspondent) without a reference to the mathematical qualities of the great man, or rather of his profile. The late Professor Silvanus Thompson, F.R.S., once took the curve presented by the profile of the face of Chamberlain, as depicted in one of "F.C.G.'s" cartoons, and treated it as a half-period of a periodic curve, which he analysed. Any student familiar with harmonic analysis can obtain the cartoon for himself from the following expression:

$$58{\cdot}9 \sin A + 1{\cdot}3 \sin 3A + 9{\cdot}3 \sin 5A - 1{\cdot}4 \sin 7A + 2{\cdot}7 \sin 9A - 3{\cdot}2 \sin 11A$$
$$+ 0{\cdot}7 \sin 13A - 1{\cdot}6 \sin 15A + 2{\cdot}5 \sin 17A + 3{\cdot}4 \cos A + 2{\cdot}9 \cos 3A$$
$$- 5{\cdot}4 \cos 5A + 2{\cdot}5 \cos 7A - 4 \cos 9A - 1{\cdot}1 \cos 11A + 0{\cdot}4 \cos 13A$$
$$+ 2{\cdot}4 \cos 15A - 1{\cdot}2 \cos 17A.$$

Could there be a more forcible expression of the man's greatness? (he asks in conclusion).

Not having our "trig" tables handy at the moment we are prepared to take our correspondent's word for it. But this method of drawing a portrait in numbers certainly opens up curious possibilities. If the great Chamberlain's distinguished profile could be reduced to a mere mathematical expression, so presumably could the rather less distinguished outline of anybody else. And since no two profiles can be exactly alike, none of the resulting mathematical expressions would be exactly alike. The police would thus be given a useful complement to the finger-print system of identification, while if the automatic telephone machines could be trained to do trigonometry—not much more, after all, than a detail improvement on their existing performances—the world would have the absolutely fool-proof telephone system, in which no call would ever be missed and nobody would ever be given a wrong number. If Major Tryon's department ever finds time hanging heavy on its hands it might look into this.—*Manchester Guardian*, July 14, 1936. [Per Mr. J. Lister.]

THE PARTITION OF CUBES.

By A. Russell and C. E. Gwyther.

Mensuration involves the construction of measures and the methods of using them and the calculation of various quantities from the simplest possible measurements. It is in this last sense that the term is usually employed, and herein problems in heights and distances are solved, the lengths of curves and the areas of plane or other figures and the volumes and surfaces of solids are measured.

Once such measures are determined other considerations often arise, such as the proportion between the measures of similar bodies in terms of some corresponding part. In a world of three dimensions the third degree, which gives, for example, the measurements of volumes in the unit of length, is of great importance. And the ratios between cubes of numbers, the equivalence of the sums of cubes and suchlike problems assume an interest. For example, the volumes of similar solids, such as spheres, can be related to one another ; and, if their density is the same, the masses can be so related. While such matters may not present any great mathematical difficulty, integral solutions always have their special value, and formulae cannot always be devised to give all possible integral solutions.

It is known that any number, if sufficiently large, can be expressed as the sum of nine (*or less*) cubes of integers. It follows, therefore, that the cube of an integer can be so expressed. But the words "or less" are the interesting ones, and the desire is to express a cube as the sum of the fewest number of cubes. A well-known theorem of Fermat states that a cube cannot be expressed as the sum of two cubes ; but some cubes can be expressed as the sum of three cubes, and in some cases this can be done in more than one way. In the Table given below an attempt has been made to give a list of all the numbers between 1 and 200 the cubes of which can be expressed as the sum of three cubes.

The practical applications of the identity $d^3=a^3+b^3+c^3$ are manifold. For example, the volume of a sphere whose diameter is d is exactly equal to the sum of the three spheres whose diameters are a, b and c respectively ; or the volume of a shell, whose external and internal diameters are d and a respectively, is equal to the sum of the spheres whose diameters are b and c respectively ; and so on. It is, of course, clear that the identity

$$(nd)^3=(na)^3+(nb)^3+(nc)^3$$

follows from the former identity.

It is interesting to note that, when a, b, c and d are integers and $d^3=a^3+b^3+c^3$, $a+b+c-d$ is always a multiple of 6. While this fact is not of any use in finding such identities, it does serve as a useful check when an identity has been found.

TABLE.

[$d = (a, b, c)$ is written for the identity $d^3 = a^3 + b^3 + c^3$.]

$6 = (3, 4, 5)$
$9 = (1, 6, 8)$
$12 = (6, 8, 10)$
$18 = (2, 12, 16)$
$= (9, 12, 15)$
$19 = (3, 10, 18)$
$20 = (7, 14, 17)$
$24 = (12, 16, 20)$
$25 = (4, 17, 22)$
$27 = (3, 18, 24)$
$28 = (18, 19, 21)$
$29 = (11, 15, 27)$
$30 = (15, 20, 25)$
$36 = (4, 24, 32)$
$= (18, 24, 30)$
$38 = (6, 20, 36)$
$40 = (14, 28, 34)$
$41 = (2, 17, 40)$
$= (6, 32, 33)$
$42 = (21, 28, 35)$
$44 = (16, 23, 41)$
$45 = (5, 30, 40)$
$46 = (3, 36, 37)$
$= (27, 30, 37)$
$48 = (24, 32, 40)$
$50 = (8, 34, 44)$
$53 = (29, 34, 44)$
$54 = (6, 36, 48)$
$= (12, 19, 53)$
$= (27, 36, 45)$
$56 = (36, 38, 42)$
$57 = (9, 30, 54)$
$58 = (15, 42, 49)$
$= (22, 30, 54)$
$60 = (21, 42, 51)$
$= (30, 40, 50)$
$63 = (7, 42, 56)$
$66 = (33, 44, 55)$
$67 = (22, 51, 54)$
$69 = (36, 38, 61)$
$70 = (7, 54, 57)$
$71 = (14, 23, 70)$
$72 = (8, 48, 64)$
$= (34, 39, 65)$
$= (36, 48, 60)$
$75 = (12, 51, 66)$
$= (38, 43, 66)$
$76 = (12, 40, 72)$
$= (31, 33, 72)$
$78 = (39, 52, 65)$
$80 = (28, 56, 68)$
$81 = (9, 54, 72)$
$= (25, 48, 74)$
$82 = (4, 34, 80)$
$= (12, 64, 66)$
$= (19, 60, 69)$
$84 = (28, 53, 75)$
$= (42, 56, 70)$
$= (54, 57, 63)$
$85 = (50, 61, 64)$
$87 = (20, 54, 79)$
$= (26, 55, 78)$
$= (33, 45, 81)$
$= (38, 48, 79)$
$88 = (21, 43, 84)$
$= (25, 31, 86)$
$= (32, 46, 82)$
$89 = (17, 40, 86)$
$90 = (10, 60, 80)$
$= (25, 38, 87)$
$= (45, 60, 75)$
$= (58, 59, 69)$
$92 = (6, 72, 74)$
$= (54, 60, 74)$
$93 = (32, 54, 85)$
$95 = (15, 50, 90)$
$96 = (19, 53, 90)$
$= (48, 64, 80)$
$97 = (45, 69, 79)$
$99 = (11, 66, 88)$
$100 = (16, 68, 88)$
$= (35, 70, 85)$
$102 = (51, 68, 85)$
$103 = (12, 31, 102)$
$105 = (33, 70, 92)$
$106 = (58, 68, 88)$
$108 = (12, 72, 96)$
$= (13, 51, 104)$
$= (15, 82, 89)$
$= (24, 38, 106)$
$= (54, 72, 90)$
$110 = (29, 75, 96)$
$111 = (16, 47, 108)$
$112 = (72, 76, 84)$
$113 = (50, 74, 97)$
$114 = (18, 60, 108)$
$= (57, 76, 95)$
$115 = (3, 34, 114)$
$116 = (23, 86, 97)$
$= (30, 84, 98)$
$= (44, 60, 108)$
$117 = (13, 78, 104)$
$120 = (9, 55, 116)$
$= (42, 84, 102)$
$= (60, 80, 100)$
$121 = (49, 84, 102)$
$122 = (19, 92, 101)$
$123 = (6, 51, 120)$
$= (18, 96, 99)$
$= (44, 51, 118)$
$125 = (20, 85, 110)$
$126 = (14, 84, 112)$
$= (23, 94, 105)$
$= (63, 84, 105)$
$127 = (13, 65, 121)$
$129 = (38, 57, 124)$
$132 = (5, 76, 123)$
$= (48, 69, 123)$
$= (66, 88, 110)$
$133 = (21, 70, 126)$
$134 = (44, 102, 108)$
$= (86, 95, 97)$
$135 = (15, 90, 120)$
$137 = (44, 73, 128)$
$138 = (9, 108, 111)$
$= (69, 92, 115)$
$= (72, 76, 122)$
$= (81, 90, 111)$
$139 = (94, 96, 99)$
$140 = (14, 108, 114)$
$= (49, 98, 119)$
$= (90, 95, 105)$
$141 = (72, 85, 122)$
$142 = (28, 46, 140)$
$144 = (1, 71, 138)$
$= (16, 96, 128)$
$= (68, 78, 130)$
$= (72, 96, 120)$
$145 = (12, 81, 136)$
$= (55, 75, 135)$
$147 = (22, 75, 140)$
$150 = (24, 102, 132)$
$= (71, 73, 138)$
$= (75, 100, 125)$
$= (76, 86, 132)$
$151 = (46, 47, 148)$
$152 = (24, 80, 144)$
$= (62, 66, 144)$
$153 = (17, 102, 136)$
$156 = (65, 87, 142)$
$= (78, 104, 130)$
$159 = (3, 121, 131)$
$= (87, 102, 132)$

160 = (56, 112, 136)
= (69, 123, 124)
162 = (18, 108, 144)
= (36, 57, 159)
= (50, 96, 148)
= (59, 93, 148)
= (81, 108, 135)
164 = (8, 68, 160)
= (24, 128, 132)
= (38, 120, 138)
= (69, 99, 146)
167 = (12, 86, 159)
168 = (56, 106, 150)
= (84, 112, 140)
= (108, 114, 126)
170 = (96, 107, 141)
= (100, 122, 128)
171 = (19, 114, 152)
= (27, 90, 162)
= (54, 80, 163)
= (107, 108, 136)
172 = (1, 135, 138)
174 = (40, 108, 158)
= (45, 126, 147)
174 = (47, 97, 162)
= (52, 110, 156)
= (66, 90, 162)
= (76, 96, 158)
= (87, 116, 145)
175 = (28, 119, 154)
176 = (25, 92, 167)
= (42, 86, 168)
= (50, 62, 172)
= (64, 92, 164)
178 = (34, 80, 172)
= (48, 133, 147)
179 = (17, 57, 177)
180 = (20, 120, 160)
= (50, 76, 174)
= (63, 126, 153)
= (90, 120, 150)
= (116, 118, 138)
181 = (108, 109, 150)
184 = (12, 144, 148)
= (108, 120, 148)
185 = (57, 68, 180)
= (68, 113, 166)
186 = (18, 121, 167)
186 = (64, 108, 170)
= (93, 124, 155)
187 = (58, 131, 160)
188 = (115, 122, 149)
189 = (21, 46, 188)
= (21, 126, 168)
190 = (30, 100, 180)
= (56, 133, 163)
191 = (39, 146, 156)
192 = (34, 123, 173)
= (38, 106, 180)
= (96, 128, 160)
193 = (25, 68, 190)
194 = (90, 138, 158)
196 = (126, 133, 147)
197 = (53, 58, 194)
198 = (22, 132, 176)
= (27, 46, 197)
= (73, 135, 170)
= (99, 132, 165)
199 = (6, 127, 180)
200 = (32, 136, 176)
= (70, 140, 170)

1099. He introduced the "crouch" seat; this required a short stirrup, thus throwing the rider on to the horse's neck and relieving the horse of some of the jockey's weight.—Article on Tod Sloan in an evening paper. [Per Mr. D. F. Ferguson.]

1100. The rapid march of modern discovery renders it not only useless, but destructive, to the progress of the natural philosopher to spend much time or trouble in exploring the beginnings of his science. While he gropes about, seeking the source, his contemporaries are borne, with ever-increasing swiftness, along the broadening and deepening current of the river.—P. G. Tait. *Sketch of Thermodynamics*, 1868, p. 1.

1101. Le jour où Viète et Descartes créèrent la géométrie analytique, "la patience remplaça le génie". Rien n'est plus vrai; mais parfois, pour traiter un problème par l'analyse, il faut tant de patience, je veux dire une telle dextérité de calcul, qu'en pareil cas il est encore plus facile d'avoir un peu de génie.—E. A. Fouët, *Leçons de Géométrie Elémentaire*, p. ix (1924).

1102. Every method of research creates its own applications: thus analytical geometry is a different science from synthetic geometry, and both these sciences are different from modern projective geometry. Many propositions are identical in all three sciences, and the general subject-matter, space, is the same throughout. But it would be a serious mistake in the development of one of the three merely to take a list of the propositions as they occur in the others, and to endeavour to prove them by the methods of the one in hand. Some propositions could only be proved with great difficulty, some could hardly even be stated.—A. N. Whitehead, *Universal Algebra*, I, p. viii (1898).

NON-ASSOCIATE POWERS AND A FUNCTIONAL EQUATION.

By I. M. H. Etherington.

Several writers* have studied algebras in which multiplication is non-associative, that is, $x \,.\, yz \neq xy \,.\, z$. It is necessary† in a non-associative algebra to distinguish the possible interpretations of a power x^n. In a non-commutative non-associative algebra x^2 is unique ; x^3 can mean xx^2 or x^2x ; x^4 can mean $x \,.\, xx^2$, $x \,.\, x^2x$, x^2x^2, $xx^2 \,.\, x$ or $x^2x \,.\, x$; x^5 has 14 interpretations ; x^6 has 42 ; and so on. In a commutative non-associative algebra, the possible interpretations are fewer : x^3 is unique ; x^4 can mean xx^3 or x^2x^2 ; x^5 can mean $x \,.\, xx^3$, $x \,.\, x^2x^2$ or x^2x^3 ; x^6 has 6 interpretations ; and so on. The problem considered here is : how many meanings are there for x^n (A) in a general non-commutative non-associative algebra ? (B) in a general commutative non-associative algebra ? The answer to (A) is $\dfrac{2(2n-3)!}{n!(n-2)!}$. I am not able to find any such simple formula for (B).

It may be observed that (A) is equivalent to asking how many meanings there are in general in ordinary algebra for $x_1 : x_2 : \ldots : x_n$, where each colon stands for any non-commutative non-associative process, such as subtraction, division or exponentiation.

(A) *The number of interpretations of x^n in a general non-commutative non-associative algebra.*

Let a_n denote the required number of interpretations. Since x^n can arise in any of the ways xx^{n-1}, x^2x^{n-2}, x^3x^{n-3}, ... , $x^{n-1}x$, we have at once for $n>1$

$$a_n = a_1a_{n-1} + a_2a_{n-2} + a_3a_{n-3} + \ldots + a_{n-1}a_1.$$

The left-hand side is the coefficient of x^n in

$$F(x) = a_1x + a_2x^2 + a_3x^3 + \ldots \textit{ ad inf.} ;$$

while the right-hand side, for $n>1$, is the coefficient of x^n in $\{F(x)\}^2$. Therefore for sufficiently small values of x (assuming the absolute convergence of the series for $F(x)$)

$$F(x) = x + \{F(x)\}^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

whence $$F(x) = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1-4x}.$$

The minus sign must be taken because $F(0)=0$. So

$$F(x) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4x} = \tfrac{1}{2} - \tfrac{1}{2}\left(1 - 2x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!} \,.\, 16x^2 - \ldots\right), \quad \ldots\ldots (2)$$

the assumption of absolute convergence being justified if $|x| < \frac{1}{4}$.

* *E.g.* De Morgan, *Trans. Camb. Phil. Soc.*, *8* (1844), 241 ; Dickson, *Trans. Amer. Math. Soc.*, *13* (1912), 60. Jordan, *Göttingen Nachr.*, 1932, 569 ; 1933, 209, deals with an application to Quantum Mechanics. A forthcoming paper by the author on Genetics uses commutative non-associative algebras.

† Though not in the algebras considered by Jordan.

Hence a_n = coefficient of x^n

$$= -\frac{1}{2 \cdot n!} \cdot \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \ldots \left(-\frac{2n-3}{2}\right) \cdot (-4)^n.$$

If $n>1$, this $= [1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)]2^{n-1} \div n! = \dfrac{2(2n-3)!}{n!\,(n-2)!}$,

a formula which is easily checked for small values of n.

(B) *An attempt to find the number of interpretations of x^n in a general commutative non-associative algebra.*

The attempt fails because the general term of the expansion corresponding to (2) cannot be written down. However, as the functional equation (4) which replaces (1) is interesting on its own account, it is perhaps worth while giving the details.

Let b_n denote the required number of interpretations, and let

$$f(x) = -1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots \text{ ad inf.} \quad \ldots\ldots\ldots\ldots (3)$$

The insertion of -1 simplifies the functional equation which arises.

Since $b_n < a_n$, the series is absolutely convergent when $|x| < \frac{1}{4}$; so $f(x)$ may be formally squared.

Now x^n can be formed in any of the ways $xx^{n-1}, x^2x^{n-2}, x^3x^{n-3}, \ldots$. This series terminates with $x^{\frac{n-1}{2}} x^{\frac{n+1}{2}}$ if n is odd and >1, with $x^{\frac{n}{2}} x^{\frac{n}{2}}$ if n is even. Hence, for $n>1$,

$$\begin{aligned} b_{2n-1} &= b_1 b_{2n-2} + b_2 b_{2n-3} + \ldots + b_{n-1} b_n \\ &= \text{half the coefficient of } x^{2n-1} \text{ in } \{f(x)+1\}^2; \\ b_{2n} &= b_1 b_{2n-1} + b_2 b_{2n-2} + \ldots + b_{n-1} b_{n+1} + \tfrac{1}{2} b_n (b_n + 1) \\ &= \text{half the coefficient of } x^{2n} \text{ in } \{f(x)+1\}^2 + f(x^2). \end{aligned}$$

So the expansions of the functions $f(x)$ and $\frac{1}{2}[\{f(x)+1\}^2 + f(x^2)]$ agree from the terms in x^3 onwards. These expansions begin respectively $-1 + x + x^2 + x^3 + \ldots$ and $-\frac{1}{2} + x^2 + x^3 + \ldots$. It follows that

$$f(x) + 1 - x = \tfrac{1}{2}[\{f(x)+1\}^2 + f(x^2)] + \tfrac{1}{2},$$

that is,
$$\{f(x)\}^2 + f(x^2) + 2x = 0, \quad \ldots\ldots\ldots\ldots (4)$$

or
$$f(x) = -\sqrt{-2x - f(x^2)}. \quad \ldots\ldots\ldots\ldots (5)$$

When x is small, we know from (3) that, to a first approximation, $f(x) = -1 = f(x^2)$; substituting this in the right-hand side of (5), we get the second approximation $f(x) = -\sqrt{-2x+1}$, $f(x^2) = -\sqrt{-2x^2+1}$; similarly the third approximation is $f(x) = -\sqrt{-2x + \sqrt{-2x^2+1}}$, and so on. Hence

$$f(x) = \lim_{N \to \infty} -\sqrt{-2x + \sqrt{-2x^2 + \sqrt{-2x^4 + \ldots + \sqrt{-2x^{2^N}+1}}}}, \quad \ldots\ldots (6)$$

where each $\sqrt{}$ sign covers all that follows it.

Abandoning the attempt to find b_n explicitly by expanding this generating function, let us consider the function itself and the various other functions which satisfy equation (4) but not (3).

Suppose we assign an arbitrary value to $f(x_0)$. Then from (4) we can calculate in succession $f(x_0^2), f(x_0^4), f(x_0^8), \ldots$; and the (possibly infinite) limit of this sequence gives either $f(0)$ or $f(\infty)$ according as $|x_0| <$ or > 1. This shows that the functions which satisfy (4) are of two kinds, those for which $f(0)$ exists and those for which $f(\infty)$ exists. In fact it will be found that no curve of the system $y=f(x)$ crosses the boundaries $x=\pm 1$.

Consider first the functions for which $f(0)$ exists. These are of three types : for (4) gives $\{f(0)\}^2+f(0)=0$, so that

$$f(0)=0, \ -1 \text{ or } -\infty . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

When x is small, let the first approximation be $f(x)=kx^\alpha$.

Substituting in (5), we get as the second approximation

$$f(x)=-\sqrt{-2x-kx^{2\alpha}}, \quad f(x^2)=-\sqrt{-2x^2-kx^{4\alpha}}.$$

For $f(x^2)$ to be real the term in $x^{4\alpha}$ must predominate, so $\alpha<\frac{1}{2}$; and in order that the two approximations for $f(x)$ should agree, $k=-1$. Iterating as before,

$$f(x)=\lim_{N\to\infty} -\sqrt{-2x}+\sqrt{-2x^2}+\sqrt{-2x^4}+\ldots+\sqrt{-2x^{2^N}}+x^{2^{N+1}\alpha},\ldots(8)$$

where each $\sqrt{}$ sign covers all that follows it. The three types (7) correspond to $\frac{1}{2}>\alpha>0$, $\alpha=0$, $\alpha<0$; and $\alpha=0$ gives (6).

The functions for which $f(\infty)$ exists are found in precisely the same way. Assuming $f(x)\sim kx^\alpha$ for large x, we reach by iteration the same formula (8), but now $\alpha>\frac{1}{2}$.

If $x>0$, the right-hand side of (8) has a factor $\sqrt{x}$. Writing $f(x)=Y\sqrt{x}$, the formula becomes

$$Y=-\lim_{N\to\infty} \theta^{N+1}(x^{2^{N+1}\alpha-2^N}), \quad \ldots\ldots\ldots\ldots\ldots(9a)$$

where θ is the operation defined by $\theta t=\sqrt{-2+t}$. Inverting (9a),

$$x=\lim_{N\to\infty} (\phi^{N+1}Y)^{\frac{1}{2\alpha-1}\cdot\frac{1}{2^N}}, \quad \ldots\ldots\ldots\ldots\ldots(10a)$$

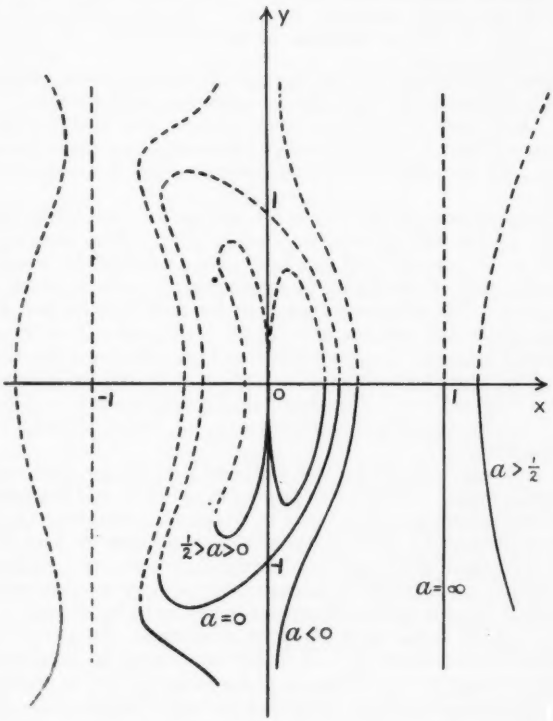
where $\phi t=t^2+2$. Similarly for $x<0$, writing $f(x)=Z\sqrt{-x}$,

$$Z=-\lim_{N\to\infty} \sqrt{2+\theta^N x^{2^{N+1}\alpha-2^N}}, \quad \ldots\ldots\ldots\ldots\ldots(9b)$$

$$x=\lim_{N\to\infty} \{\phi^N(Z^2-2)\}^{\frac{1}{2\alpha-1}\cdot\frac{1}{2^N}}. \quad \ldots\ldots\ldots\ldots\ldots(10b)$$

The curves $y=f(x)$ can be traced from these formulae (8)-(10) combined with (4). The various types are shown by the continuous lines in the figure. When $\alpha<0$ or $>\frac{1}{2}$, the curve has two branches. If $f(x^2)$ in (4) is taken to mean " one value of $f(x^2)$ ", then all the radicals in (8) and (9) can be taken as " plus or minus ". (Most of the radicals have to be taken positively to yield a real value of $f(x)$.)

With this generalisation the curves continue as shown by the dotted lines in the figure. The zeros of $f(x)$ can be calculated from equations (10) by putting $Y=0$, $Z=0$. These equations also show that



for $\alpha=\infty$, $x=\pm 1$; and that for $\alpha=\frac{1}{2}$ the curve becomes the line at infinity plus the origin. Actually, when $\alpha=\frac{1}{2}$, the process of iteration terminates and gives the solution $f(x)=kx^{\frac{1}{2}}$, where k is one of the (imaginary) roots of $k^2+k+2=0$. I. M. H. E.

1103. . . . We have had many earlier promises that the Government needed more time for the consideration of Mr. S——'s proposals—time that has been infinite. (In fact, it has had the latest proposals before it for nearly four months.)—*Manchester Guardian Weekly*, November 20, 1936, p. 402, c. 3 [Per Mr. J. W. Ashby Smith.]

THE BOARD OF EDUCATION SUMMER SCHOOL FOR TEACHERS OF MATHEMATICS IN TECHNICAL COLLEGES, JULY, 1936.

By a Member of the Course.

For about fourteen years the Board of Education has held summer schools at Oxford for teachers of mechanical and electrical engineering. Last year, for the first time, a course was held for teachers of mathematics. It was held at the same time as the engineering courses, and members of all the courses lived together in New College.

As many readers of the *Gazette* are unfamiliar with these courses, it would be as well to outline what is done. The mornings are devoted to two lectures of one hour each, followed by about one hour and a half of discussion in small groups, each of which is led by a tutor. The afternoons are free for amusement, and in the evenings there are popular lectures on topics related to the work of technical colleges. Last year these evening lectures were common to the three courses, mechanical, electrical and mathematical; they included lectures on "The Mind of Primitive Man", "Frameworks", "Engineering Developments in Soviet Russia" and "Gyro-actions".

The morning lectures in the mathematical course had as their underlying theme, "How to improve the syllabus and presentation of the mathematics of the ordinary National Certificate Course". The most important of these lectures was a course of four by Mr. Verity, Principal of Sunderland Technical College, on "Methods of Presentation". In these he stressed the importance of introducing new functions and their differential coefficients by numerical and graphical work before any rigorous treatment. In particular he emphasised the introduction of fractional powers by points on the smooth curve given by values of 2^x for integral x; the calculation of the approximate rate of change of a simple function like x^3 for a large number of values of x, and plotting these values against x^2 to show that the rate of increase of x^3 is $3x^2$; the plotting of the gradient of 2^x against 2^x to show that they have a constant ratio, and the introduction of e as the number which makes this ratio unity for the function e^x. To do this kind of work properly involves a great deal of calculation, and Mr. Verity pressed for the division of each class into teams, each member of a team calculating one value of a function or its gradient. He also advocated drawing the graphs of y/x, xy, dy/dx, $\int y\,dx$, etc., from the graph of y by graphical methods.

To begin with, most of us felt that the graphical approach had been overstressed in these lectures, but in a discussion later in the course Mr. Verity made it clear that he considered the graphical methods should never be used to the exclusion of the analytical;

nevertheless he thought that they should always precede and lead up to the analytical methods.

The other lectures were: "The Syllabus of Mathematics in relation to Engineering Science", by Dr. Abbott, H.M.I., who dealt with a large number of points, his main one being that the engineering student, because of his profession, was not likely to be interested at first in the philosophical side of mathematics so we must catch his interest early, and later, in order to show him the necessity for logical proofs, we might trip him up by some easy mathematical fallacies; the chief reason, he thought, for introducing the calculus early was that it helped to get the student interested. "The Marking and Setting of Papers", which was a very frank talk by two assessors of National Certificate papers. "The Misuse of Mathematics", by Mr. Stelfox, H.M.I., who urged us not to use mathematics where plain English was better. "The Use of Graphical Methods", by Dr. Stuart, Headmaster of Hackney Junior Technical School, an excellent lecture, which was really on how to weave all the methods at the disposal of an elementary student into working out a problem within his reach; Dr. Stuart's chief example was taken from the specification of an Austin Seven, which he used to calculate the average pressure on the piston from the horse-power, the turning moment on the crank from the pressure on the piston, and so back to the horse-power from the area under the turning moment—crank angle graph; an example involving plotting of graphs, trigonometry, and integration both by exact and graphical methods, and one of the type we should all like to have time to do. "Elementary Algebra in Secondary Schools", by Mr. Parker, H.M.I., who gave statistics indicating general weakness in manipulation. "Mathematics in the Engineering Industry", by Mr. Wheeldon, of Metropolitan-Vickers at Salford, who deputised for Mr. Bailey and dealt with methods of successive approximation used in solving the differential equations of stress in engineering parts of various shapes.

I have left to the end three lectures which were taken by members of the mathematics course, and of one or both of the engineering courses. They were: "Workshop Mathematics", by Dr. Fisher, Principal of Wolverhampton Technical College; "Films on the Mathematical Aspects of Engineering", by Mr. Dance, H.M.I.; "The Working of Examples in Mechanics", by Professor Haigh, Royal Naval College, Greenwich. The last of these was so outstanding that I must give an outline of it here. Professor Haigh divided the working of an example into four parts. (1) Practical—state the problem or question, also what is to be taken into account and what neglected. (2) Mechanics—state the suitable laws of nature in their applicable forms, and write down the equations derived from them. (3) Mathematics—If (1) and (2) are true and applicable, then .... (4) Arithmetic—substitute values for symbols in the results of (3). Teachers, he said, usually put down (1), but we should make the students find problems suitable for the work in

hand ; (2) is usually covered by lectures ; we should start with easy examples that jump straight from (2) to (4). Professor Haigh then worked out the following problem : " Investigate the oscillation of a belt in an engine test (brake test), taking account of friction and neglecting the mass of the rope " ; and the way in which he put down the sections falling under headings (2) and (3) was the method which most mathematical teachers adopt, but, one felt, few engineers ; nevertheless the engineer has this great advantage, that he usually states (1), the practical problem, far more clearly than the mathematician.

We also had two general discussions : one amongst mathematical teachers only, which followed the lines of Mr. Verity and Dr. Abbott's lectures. Many points were raised in this discussion, but I think the outstanding ones were : (1) the necessity for any new function, such as the sine or cosine, should be shown by means of a practical problem ; (2) examples should be real, and not of the type " The combined ages, etc. ", and that they should be taken from engineering only where elementary examples were available ; (3) evening students should have constant revision of the previous year's work by means of short tests. The other general discussion was between members of all three courses on " Cooperation between the Departments of Mathematics and Engineering ". It was opened by Mr. Geary and Dr. Docherty, of Northampton Polytechnic. Mr. Geary made a strong plea for mathematics classes being taught by professional mathematicians and not by engineers, but he said that their work would benefit greatly if engineers would hand on to them examples of a mathematical nature from modern engineering practice.

I believe the general opinion of members of the course was that it was a good beginning. I am sure that many attending the mathematics course would have liked to have been free to go to some lectures in the engineering courses. If I may venture to make a suggestion or two for future years, they are (1) that part of the mathematics course should consist of fairly elementary lectures in engineering, accompanied by practical demonstration in engineering shops if possible, and, conversely, that the engineering courses should have some lectures on mathematics ; (2) there should be one or two lectures on advanced mathematical methods which have applications in engineering.

In the above account I have inevitably given more prominence to the lectures which interested me most, or which were easier to report. I hope it will, at any rate, remind readers of the *Gazette* whose work is in public or secondary schools of the enormous number of boys who are studying mathematics at polytechnics. I believe that our problem in technical colleges is their problem too, for, except in a very few cases, boys in secondary schools do not become mathematical specialists. If boys are to be got to look at the problems of daily life from a mathematical, or analytical, point of view they must first be interested in the necessity for using

mathematics, and so mathematics must be for them an experimental science to begin with, growing later on into a theoretical and philosophical study. We should all try to get more boys up to the "If" stage of Professor Haigh; that is, to get them able to express their ideas in mathematical symbols, then they will see the necessity for algebraic manipulation and enjoy it. Algebra, by itself, is the dry bones of mathematics to such boys, into which we have to bring life by means of Professor Haigh's first two stages, but, while cutting down the amount of difficult algebra to begin with, we must never let our work degenerate into the memorising of formulas and arithmetical substitution. It is well that we should realise that the faculty of being able to connect up the four stages, into which Professor Haigh divided his examples, for difficult problems in engineering or physics is far less common throughout the country than the faculty of being able to investigate an obscure point in the theory of numbers.

1104. High winds have harassed air line pilots recently. But as one of them pointed out to me yesterday, they benefit from Newton's Third Law. By this it followed that upon occasions they were travelling with the wind.

In support of his argument he quoted various recent instances of very rapid flights by British commercial aircraft.

Here are three of them:

Newtownards–Isle of Man (55 miles), 15 min.
Isle of Man–Liverpool (90 miles), 30 min.
Jersey–Heston (182 miles), 60 min.

The quickest has an average of 220 M.P.H., the slowest of 180.—*Daily Telegraph*, October 22, 1935.

1105. HEAVY-GOING HUMOUR.

Possibly there will have been some people in Oxford and elsewhere who will have been sorry to hear Mr. A. P. Herbert's announcement of the extreme seriousness of his candidature, for Oxford University elections have produced clouds of squibs, good and bad, in their long history, and those who have taken part in that game have included some famous humorists, notably "Lewis Carroll". Dodgson's effort on the occasion of the Hardy-Gladstone fight seventy years ago would probably be regarded as rather tough stuff in the way of humour by most people to-day. It was called "The Dynamics of a Parti-cle". It contained such passages as this:

On Differentiation. . . . The effect of Differentiation on a Particle is very remarkable, the first Differential being frequently of greater value than the original particle and the second of less enlightenment. For example, let L equal "Leader", S equal "Saturday", and then LS equals "Leader in the Saturday" (a particle of no assignable value). Differentiating once we get L.S.D., a function of great value. Similarly it will be found that by taking the second Differential of an enlightened particle—*i.e.* raising it to the Degree D.D.—the enlightenment becomes rapidly less.

And much more which suggests that our grandfathers had a more massive sense of humour than ours.—*Manchester Guardian*, November 8, 1935.

1106. FROM A SCRIPT.

$\sin P/2 = 3.05/3$. The result is imaginary probably in the fourth dimension. [Per Mr. C. E. Kemp.]

"NATURAL" MATHEMATICS.

By M. S. Bartlett.

As a science becomes mature it usually relies more and more on mathematical ideas. It has followed that in so far as mathematics is not entirely a "pure" subject, its content has been dictated by the more mature, that is, by the physical sciences. Nevertheless, it is slowly being appreciated that the permeation of mathematics through to the non-physical sciences must imply sooner or later a partial reorientation of the existing attitude.

For one recent tendency in algebra for the teaching of matrix theory to be advocated at a less advanced stage, it is probable that the attention given to this theory since its introduction into physics has been largely responsible. It seems worth pointing out, however, that the same reason for this introduction, namely, the use of matrix theory in a statistical theory of observation and measurement, will be cooperating towards its use in other fields. In genetics, for example, the probable set of genotypes resulting from a specified method of artificial selection for n generations on the original vector set U_0 is represented by $U_n = S^n U_0$, where S is the selection matrix. In psychology, factor theories of ability postulate that an observed set of test scores for a number of persons will depend on an equation of the form $T = MF$, where F is the matrix of factors for the persons, and T the matrix of observed scores.

Since statistical theory and method is my own subject, I naturally cannot resist stressing this association of such applications of matrix theory with a calculus of statistics and observation. I might further take the opportunity to point out that while the universities may be beginning slowly to recognise the need of statistics in the curriculum not as an advanced mathematical subject, but almost as a necessity for workers in any of the observational sciences, the value of beginning the teaching of statistical ideas earlier, before the university stage is reached, is apparently not, so far, accepted. Yet if we recognise that statistical method is often the only possible numerical method in some of the newer sciences, a development of mathematical teaching to meet their needs would seem to demand just this subject. Irresponsible handling of statistics has given it critics, but to these it should be pointed out that a better use of a method should follow from a better understanding.

My chief concern here, however, is not to advocate *what* it is that intermediate students who may need to use mathematics in, say, the biological or social sciences should be taught, but *how* they are to be taught. It seems to me that many of these students might be helped if a more pragmatic view of mathematics as they require it were adopted. I have the highest regard for the subject of pure mathematics, but it does not necessarily follow that the students I have in mind will benefit most from an extremely pure mathematical standpoint.

It is possible that, as a contrast to the tautological nature of pure

mathematics, they would benefit by regarding their mathematics almost as a physical theory of counting and measurement. An important aspect of this approach is that the physical viewpoint is not confined to what is more commonly regarded as " applied " mathematics, for the methods applicable to the younger sciences may be sufficiently undifferentiated to be based on general principles which, often confusingly to the learner, are classed as " pure " mathematics. Thus it would no doubt greatly assist learners of probability theory, which is the basis of statistical method, if they were told that probability, as it is used in genetics, experimental psychology, physics, actuarial science and modern statistical inference, is a " frequency theory " about physical events, and not be taught to regard it either *merely* as a branch of algebra or even as an excuse for mysticism.

Even the more classical branches of mathematics do not suffer from this naturalistic point of view. In elementary geometrical teaching it is, I believe, already the tendency to place propositions on a physical basis by practical mensuration, so that it may be grasped that certain theorems, though they follow from certain axioms which may not be true, appear sufficiently valid under certain practically realisable conditions.

It is interesting to consider the infinitesimal calculus from this position. When this calculus was first introduced by Newton and Leibnitz there was no reason to suppose that it could not exactly mirror reality ; but those days are now gone. In practice all intervals are finite for convenience, and all real integration might justifiably be regarded as of a Stieltjes, not of a Riemann, type. It is instructive to realise that whereas to the pure mathematician summation is an approximation to integration, in reality integration is an approximation to summation. This difference would not be important if it were one merely of practical convenience and could be reduced indefinitely, but the present outlook is that of a discontinuous and statistical framework in which the possible reduction is limited. We shall certainly not give up the powerful method of the infinitesimal calculus merely because it may not have an exact counterpart in nature, but we are not justified in regarding its predictions as more than most other mathematical results when applied to the real world—as numerical approximations to the truth.

In a similar way the possible occurrence of infinity in an equation is not always an inevitable mathematical drawback to the use of this equation as an approximation over a wide range to a physically more exact theory. The user will be less perturbed by any pure mathematical lack of rigour in his equation than to the signs that, in so far as infinity (or an infinitesimal) can occur, his equation may for certain conditions not only be physically inexact but no longer even a good approximation ; for there is no evidence that infinity and infinitesimals occur in nature, only relations of large and small.

M. S. B.

ON CERTAIN TRANSFORMATIONS IN EUCLIDEAN SPACES.

By E. A. Maxwell.

This paper attempts to illustrate the method of representation, which is so powerful a weapon of modern geometry, in terms of elementary Higher Certificate mathematics. The results in their general form are familiar to students of geometry; but the special problem given here has several interesting features of its own.

No originality is claimed for this paper, though I do not know of any other account of the work. It seems, however, worth while to give a brief summary, as it covers a great deal of the ground with which schoolboys should be familiar.

The results are given for pairs of points and for circles. It will be obvious (and indeed it forms an interesting exercise) that the work can be extended to spheres and to corresponding figures in higher space.

I. *Pairs of points on a given straight line.*

1. A pair of points upon a given straight line l may be defined in position by the pair of roots of the quadratic equation

$$x^2+2px+q=0,$$

where (if necessary) x may be regarded as the distance of a member of the pair from some fixed point.

We note at once that a pair is uniquely defined if p and q are given; and, conversely, p and q are uniquely defined by the pair of points on l. Hence if p and q are chosen as cartesian coordinates in a plane ϖ, then to a point of ϖ corresponds a unique pair on l, and to a pair on l, conversely, corresponds a unique point in ϖ.

We have thus set up a (1, 1) *correspondence between the pairs of points on a line l and the points of a plane* ϖ.

2. We may, if we wish, regard a single point α of l as a coincident pair whose equation is

$$x^2-2\alpha x+\alpha^2=0,$$

so that

$$p=-\alpha\,; \quad q=\alpha^2.$$

It follows that *the single points of the line l can be considered as represented birationally upon the parabola* Ω *whose equation is* $p^2=q$ *in the plane* ϖ.

3. The study of involutions is very conveniently pursued by means of this representation in the plane. For consider the involution determined by the two pairs

$$x^2+2p_1x+q_1=0,$$

$$x^2+2p_2x+q_2=0.$$

Any pair of the involution is

$$\lambda_1(x^2+2p_1x+q_1)+\lambda_2(x^2+2p_2x+q_2)=0,$$

and so is represented by the point

$$p=\frac{\lambda_1 p_1+\lambda_2 p_2}{\lambda_1+\lambda_2}; \quad q=\frac{\lambda_1 q_1+\lambda_2 q_2}{\lambda_1+\lambda_2},$$

which lies upon the line joining (p_1q_1) and (p_2q_2).

The pairs of points of an involution are thus represented in ϖ by points lying upon a straight line.

4. Certain deductions follow rapidly from the preceding paragraph. Firstly, since the straight line representing an involution cuts the parabola Ω in two points, it follows that *the involution has two double points*, which may coincide if the representing line touches the parabola. Again, since two straight lines in the plane ϖ necessarily meet in a point, it follows that *two involutions upon the line l have one common pair.*

5. It is well known that the two pairs

$$x^2+2p_1x+q_1=0,$$
$$x^2+2p_2x+q_2=0,$$

separate each other harmonically if

$$2p_1p_2=q_1+q_2,$$

which is precisely the condition that the points (p_1q_1) and (p_2q_2) should be conjugate with respect to the parabola Ω.

We thus see that there is one pair of points which separates harmonically each of two given pairs; namely the pair represented by the intersection of the polars of the two points which represent the two given pairs. In like manner, we see that there is a unique pair of points which harmonically separates each pair of an involution: namely the pair represented by the pole of the line which represents that involution.

It may be added that, as we have seen, the individual points of the line l are represented birationally by the points of the parabola Ω. It is immediately obvious that the two points which form a pair represented in the plane ϖ by a point P are themselves severally represented by the two points in which the polar of P with respect to Ω cuts that parabola.

As an illustration of these ideas, we note that three points forming a triangle which is self-conjugate with respect to Ω represent three pairs of points, of which each pair of points separates each of the other pairs harmonically. Again, by considering the triangle self-conjugate with respect to Ω which arises from the complete quadrilateral defined by four points lying on Ω, we can prove the following result:

If A, B, C, D are four points upon a straight line, and X_1, X_2 is the pair of points harmonically separating B, C and A, D; Y_1, Y_2 and Z_1, Z_2 being similarly defined for C, A and B, D; A, B and C, D: then each of the pairs X_1, X_2; Y_1, Y_2; Z_1, Z_2 harmonically separates each of the other pairs.

II. *Circles in a given plane.*

1. A circle lying in a given plane π may be specified by the equation

$$x^2+y^2+2px+2qy+r=0.$$

This circle is uniquely defined if p, q, r are given, and, conversely, p, q, r are uniquely defined by the circle. Hence if p, q, r are chosen as cartesian coordinates in a space S of three dimensions, then to a point of S corresponds a unique circle in π; and to a circle in π, conversely, corresponds a unique point in S.

We have thus set up a (1, 1) *correspondence between the circles in a plane* π *and the points of a space* S.

2. We may, if we wish, regard a point (α, β) of the plane π as a circle of zero radius whose equation is

$$(x-\alpha)^2+(y-\beta)^2=0,$$

so that $p=-\alpha, \quad q=-\beta, \quad r=\alpha^2+\beta^2$.

It follows that the points of the plane π can be considered as represented birationally upon the paraboloid of revolution Φ whose equation is $p^2+q^2=r$ in the space S.

3. The study of coaxal systems is very conveniently pursued by means of this representation in S. For consider the coaxal system defined by the two circles

$$x^2+y^2+2p_1x+2q_1y+r_1=0,$$
$$x^2+y^2+2p_2x+2q_2y+r_2=0.$$

Any circle coaxal with these is

$$\lambda_1(x^2+y^2+2p_1x+2q_1y+r_1)+\lambda_2(x^2+y^2+2p_2x+2q_2y+r_2)=0,$$

and so is represented by the point

$$p=\frac{\lambda_1p_1+\lambda_2p_2}{\lambda_1+\lambda_2}; \quad q=\frac{\lambda_1q_1+\lambda_2q_2}{\lambda_1+\lambda_2}; \quad r=\frac{\lambda_1r_1+\lambda_2r_2}{\lambda_1+\lambda_2},$$

which lies upon the straight line joining $(p_1q_1r_1)$ and $(p_2q_2r_2)$.

The circles of a coaxal system are thus represented in S *by points lying upon a straight line.*

Since this straight line meets the paraboloid Φ in two points, it follows that there are two " limiting points " of the coaxal system, represented by the two points so obtained.

4. Two circles,

$$x^2+y^2+2p_1x+2q_1y+r_1=0,$$
$$x^2+y^2+2p_2x+2q_2y+r_2=0,$$

are known to cut orthogonally if

$$2p_1p_2+2q_1q_2=r_1+r_2,$$

which is the condition that the points $(p_1q_1r_1)$, $(p_2q_2r_2)$ should be conjugate with respect to Φ.

Hence orthogonal circles are represented by points which are conjugate with respect to the paraboloid Φ.

From this we can deduce that there is a unique circle which cuts orthogonally each of three given circles.

5. We have seen that the points of the plane π are represented birationally by points of the paraboloid Φ. We now seek those points on Φ which correspond to the *points* of a circle represented by the point (p_1, q_1, r_1) in S. To do this, we note that the point (α, β) lies upon the circle in question if

$$\alpha^2+\beta^2+2p_1\alpha+2q_1\beta+r_1=0.$$

Moreover, the point (α, β) of π is, as we have seen in § 2, itself represented by a point (p, q, r) on Φ, where

$$p=-\alpha\,; \quad q=-\beta\,; \quad r=\alpha^2+\beta^2.$$

The condition is thus

$$2pp_1+2qq_1=r+r_1,$$

which is the equation of the polar plane of $(p_1q_1r_1)$ with respect to Φ.

Hence the points of a circle $(p_1q_1r_1)$ are represented on Φ by the points in which it is met by the polar plane of $(p_1q_1r_1)$ with respect to Φ.

6. We are now in a position to study further the coaxal system represented in S by points of the line l which meets the paraboloid Φ in A and B. Let λ be the polar line of l with respect to Φ, meeting Φ in C and D. Then it is easy to verify that the circles represented by the points of λ are the members " opposite " to the given system of coaxal circles ; that each circle of the l system cuts each circle of the λ system orthogonally ; that the limiting points of the l system are represented (as we have seen) by A and B, and the λ system by C and D ; that each circle of the l system passes through the points represented by C and D, and the λ system by A and B ; so that the familiar properties of such systems are readily verified.

7. Certain other results may be stated briefly. The circle through three points represented on the paraboloid by A, B, C is itself represented by the pole of the plane ABC with respect to the paraboloid. A system of concentric circles is represented by the points of a straight line parallel to the r-axis ; and the centre of any circle is represented on Φ by the point in which it is met by the straight line drawn parallel to the r-axis, and through the point representing that circle. Two circles which touch are represented by two points whose join touches Φ, their point of contact being represented by the point of contact of their join with Φ.

From this last statement it follows that there are eight circles touching each of three given circles. For let the three circles be represented in S by the three points A, B, C ; any circle touching A is represented by a point on the tangent cone from A to Φ, similar results holding for B and C. These three tangent cones meet in eight points * which represent the eight desired circles. E. A. M.

* Geometers will note that these eight points form an " associated " set.

THE PILLORY.

School Certificate and Matriculation Examination, Universities of Manchester, Liverpool, Leeds, Sheffield and Birmingham.

MATHEMATICS (GEOMETRY), *Tuesday, July 7th*, 1936 ; Qn. 1.

" A tree is 30 ft. from a straight ditch 100 ft. long and is equidistant from the ends. Treasure is buried *between the tree and the ditch* at a point 45 ft. from the tree and 20 ft. from the ditch. Using ruler and compasses only, make a plan (scale 1 cm. to 10 ft.) showing the ditch, the tree and the possible positions of the treasure."

The italics are the examiner's. The wording appears to be ambiguous.

F. W. HENDERSON.

Law Preliminary. GEOMETRY, Qn. 5. *February* 14, 1935.

" $PQRS$ is a rectangle ; the diagonal SQ is bisected at X and AXB is drawn perpendicular to SQ so that $XA=XB=XS$. Show that if PA, BR be joined they are parallel and that the square $ASBQ$ is equal to $PQRS$ in area."

The latter statement is true if $PQRS$ is a square.

J. W. BROOKS.

London Intermediate Arts and Science (External), July 1933.

PURE MATHEMATICS. I. Qn. 7.

" Prove the following properties of two lines AB and CD which do not lie in the same plane : . . . (iv) the locus of points equidistant from AB and CD is a plane."

The locus is a hyperbolic paraboloid.

H. A. HAYDEN.

Examination Papers . . . set for Mathematical Scholarships at Merton, New College, Brasenose, Christ Church and Hertford. (Reprint.)

December 1934.

II. 6. Read " small *smooth* pulley ".

II. 7. For $\left(1+\frac{1}{e}\right)^2$ read $1+\frac{1}{e^2}$.

III. 2. (iii) For 2^{n-1} read 2^{2n-1}.

IV. 1. For $(-)^n$ read $(-)^{n-1}$.

IV. 4. For " elements " read " terms ".

V. 3. For " hy^2 " read " by^2 ".

Open Mathematical Scholarships and Exhibitions. Balliol, Queen's, Corpus Christi and St. John's. December 1933. (Reprint.)

III. 5. For " is " read " it ".

IV. 1. For " + " read " − ".

IV. 6. In the numerator of the expression for A_1 omit $-\phi(a_1)$.

A. S. GOSSET TANNER.

London University Intermediate Examination in Engineering (External Students). November 1933.

APPLIED MECHANICS.

"9. Prove that the force exerted on a plane surface by a uniform liquid in which it is immersed is $g\rho\bar{y}$, where ρ is the mass of unit volume of the liquid and $\bar{y}$ is the depth of the centroid of the area below the surface of the liquid." E. H. LOCKWOOD.

Merton, New College, Brasenose, Christ Church and Hertford. Mathematical Scholarships and Exhibitions, March 1936.

MECHANICS.

"P, Q, R, S are the mid-points of the sides AB, BC, CD, DA of a quadrilateral. Prove that

$$2(PR^2 - QS^2) + AB^2 - BC^2 + CD^2 - DA^2 = 0.$$

A linkage is formed of four straight bars AB, BC, CD, DA freely hinged at their ends. Strings PR, QS connect the mid-points of opposite bars. Show that the tensions of the strings and the thrusts in the bars are proportional to their lengths."

1. Connectors of P, R and Q, S must be one in tension and the other in thrust. (See Ramsey's *Statics*, p. 207.)

2. One pair of opposite bars are in tension, the other pair in thrust.

3. One cannot rightly speak of the tensions or thrusts in the bars, as these are different for each half of a bar.

4. The question seems to imply that the reactions at the corners are along the bars—which is not true. A. S. GOSSET TANNER.

Law Prelim. 13th February, 1935.

ARITHMETIC.

"How many bricks are needed to build a wall 12 feet high, 30 feet long and 9 inches thick with bricks measuring $9 \times 3 \times 4\frac{1}{2}$, assuming $\frac{1}{4}$ inch of mortar between bricks?"

It is presumed that the figures $9 \times 3 \times 4\frac{1}{2}$ are all in inches, but the question does not say so. J. W. BROOKS.

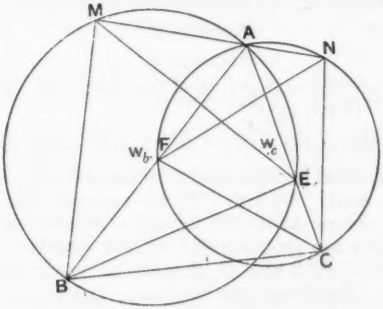
1107. Compared to the modern critical period which began in the latter part of the Nineteenth Century, the Eighteenth appears to some like a brilliantly gifted boy of ten who has wandered into an extremely dangerous machine shop. The boy is fascinated by the flashing pistons and humming wheels, the more so as he soon discovers that he can make the machinery do all sorts of spectacular stunts by pulling a switch here or depressing a lever there, and it is only by the grace of God and the presence of mind of the foreman that the young scientist is not returned to his mother in a bucket.—E. T. Bell, *The Search for Truth* (Allen and Unwin, 1935), pp. 187-8.

1108. Das neue Hans sieht kalt und "sachlich" aus, praktisch, mathematisch und unpersönlich.—O. Koischwitz, *Deutsche Fibel*, p. 3. [Per Mr. V. Naylor.]

MATHEMATICAL NOTES.

1220. *Sur le triangle isoscèle.*

1. Sur les côtés BA et AC d'un triangle quelconque ABC, marquons des points arbitraires F et E, situés d'un même côté de BC, par exemple entre A et B, A et C. Les cercles (w_b) et (w_c), circonscrits aux triangles BAE, CFA, coupent respectivement la bissectrice extérieure de l'angle A en des points M et N situés aux milieux des arcs sous-tendus par les cordes BE, CF.



Les triangles BME et CFN sont donc isoscèles ; de plus, ils sont ~~semblables~~, et leurs côtés BE et FN, CF et ME sont ~~parallèles~~, à cause des égalités d'angles *

$$MEB = MAB = 90° - \tfrac{1}{2}A = \tfrac{1}{2}(B+C) = CAN = \text{~~}FNC\text{~~}. \quad \ldots\ldots(i)$$

2. Supposons maintenant les points E, F fixés de manière que l'on ait la relation d'angles

$$CBE - FCB = \tfrac{1}{2}(B - C). \quad \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Les angles opposés CBM et MNC du quadrilatère $BMNC$ ont pour expressions

$$CBM = CBE + EBM = CBE + 90° - \tfrac{1}{2}A,$$

$$MNC = MNF + FNC = ACF + FNC = C - FCB + A,$$

et l'on a

$$\begin{aligned} CBM + MNC &= CBE - FCB + 90° + \tfrac{1}{2}A + C \\ &= \tfrac{1}{2}(B - C) + 90° + \tfrac{1}{2}A + C \\ &= \tfrac{1}{2}(A + B + C) + 90° \\ &= 180°. \end{aligned}$$

Dès lors, *le quadrilatère $BMNC$ est inscriptible dans une circonférence* (Σ).

N.B.—1°. Si l'on suppose que les droites BE et CF sont *égales*, puis, que, par hypothèse, l'égalité (ii) soit vérifiée, les triangles isos-

* *Gazette*, XVIII, p. 256.

cèles BME et FNC sont *égaux*, et $BM = CN$. Le quadrilatère inscrit $BMNC$ est un trapèze isoscèle qui conduit aux relations d'angles

$$CBE + EBM = CBM = NCB = NCF + FCB,$$

et

$$EBM = FCB.$$

Les triangles CBE, BFC sont égaux, et $B = C$. Le triangle ABC est isoscèle et on peut énoncer ce théorème :

Si deux droites égales BE, CF, situées d'un même côté de BC, coupent les côtés AC et AB d'un triangle de façon que la différence des angles CBE et FCB soit égale à celle des angles B et C du triangle, ce dernier est isoscèle.

Reciproquement, d'ailleurs, dans un triangle isoscèle ABC, si l'on trace deux droites BE et CF de manière que

$$CBE - FCB = \tfrac{1}{2}(B - C) = 0,$$

on a

$$CBE = FCB,$$

et les droites BE et CF sont égales.

3. Les bissectrices intérieures BE et CF d'un triangle quelconque ABC satisfont à la condition (ii). Par suite, le théorème qui précède généralise celui-ci :

Tout triangle qui a deux bissectrices intérieures égales, est isoscèle.

V. THÉBAULT.

1221. *Medial section.*

In connection with medial section, one of my pupils, H. T. Corke (aged 14.11), has found out two theorems which may be new to many teachers.

Denoting by $\lambda : 1$ the ratio of a line to its longer medial segment :

(i) if in an ellipse the ratio of the axes is $\lambda : 1$, then the ellipse is equal in area to the circle whose diameter is the join of the foci ;

(ii) $(\lambda + 1)$ radians $\simeq 150°$, so that $\pi \simeq \frac{3}{5}(3 + \sqrt{5})$.

The frequency with which λ occurs in mathematics is very remarkable, and those who think that it is only of interest in connection with the regular pentagon are surely mistaken.

For example, if $ABCD$ is a square and P lies on DC produced so that $AP : PB$ is a maximum, then $AP : PB = \lambda : 1$.

Again, if a series is formed so that each term is the sum of the two previous terms, then the ratio of consecutive terms of the series tends to $\lambda : 1$.

All this is apart from the question whether the perfect human figure is divided at the waist in this ratio ; those who hold this latter theory make the ratio almost ubiquitous in the human body.

C. O. TUCKEY.

1222. *On Note* 1201.

The proof that the bisector of an angle of a triangle divides the opposite side in the ratio of the other two sides, given in Note 1201, appeared in Note 734 which I contributed to the *Gazette*, October 1924. R. HAMILTON DICK.

1223. *Differentials.*

In reply to the footnote on p. 277 of the *Gazette* for October 1936, may we say that we do *not* begin with $ds^2 = dx^2 + dy^2$ and that we believe that it is both unnecessary and injurious to do so ?

We accept the intuitive idea of the length of an arc as the limit of the perimeter of an inscribed polygon, and regard it as unsuitable to discuss *necessary* conditions of rectifiability in school work.

But the idea of the " several sides becoming ultimately coincident with the respective corresponding elements of arc ", if it is to be the intuitive basis of the whole theory, will have to be translated into a language which we and our pupils can understand. A.R.
C. V. D.

1224. *The " ambiguous case " in the solution of triangles.*

Note 1177 (*Gazette*, XX, p. 55) presents in an interesting manner an entirely trigonometrical discussion of all the cases which may arise when two sides of a triangle, and an angle not included, form the data. But it often happens that the best method of presenting a theory is not necessarily the clearest way of explaining it, nor the best for purposes of calculation, and it may not be out of place here to gather up a few hints on this aspect of the subject.

When teaching the solution of triangles it is helpful to stress the fact that there is a correspondence between the geometrical and the trigonometrical contingencies that may have to be faced. Thus the first thing the student should do with the data is to attempt a rough sketch of the triangle. The angle A should be drawn first ; the side b must be determined next, and this fixes the point C.

Let AX be the line on which the point B lies.

The side a will then be attempted by describing an arc with centre C and radius a. The various contingencies then are :

If A is obtuse and $a \leqslant b$, no triangle is possible.

If A is obtuse and $a > b$, one and only one triangle is possible.

If A is acute, then the arc of radius a may

(I) miss AX altogether ;

(II) touch AX ;

(III) cut AX at two points on the same side of A ;

(IV) cut AX at two points of which A is one ;

(V) cut AX at two points on opposite sides of A.

Cases (III) to (V) are all included in case (3) of Note 1177, while cases (I) and (II) correspond respectively to cases (1) and (2) of that Note. The point here is that these cases should be discussed geometrically, as above, side by side with the trigonometrical discussion

in the corresponding part of the previous Note; the point may appear trivial, but is often overlooked and seems to be extremely helpful to beginners.

Now the above case (V) occurs when, and only when, we have $a>b$; hence, in attempting the solution of a triangle with a particular set of data, we should look at once to see whether the side opposite the given angle is greater than the other given side; if it is, then no ambiguity will arise. (This is true whether A is acute or obtuse.) If $a=b$, no triangle is possible, but if $a<b$ there is still a possibility of ambiguity and we must keep our eyes open a little longer. Note, however, that, admirable as the discussion of the sign of C is from the theoretical aspect, in practice we can ascertain in some cases before calculation commences that no ambiguity will arise.

Suppose now that we find $a<b$. We have to employ the formula $\sin B=(b \sin A)/a$. Note 1177 shows two ways of doing this; in all there are three:

$$\text{(i) } \log \sin B=\log b-\log a+\log \sin A;$$
$$\text{(ii) } \log \sin B=\log \sin A+\log b-\log a;$$
$$\text{(iii) } \log \sin B=\log b+\log \sin A-\log a.$$

Given a table of logs and log sines, it would seem that (i) requires only one turn-over, viz. from logs to log sines; (ii) requires two—from log sines to logs and back for the answer; (iii) requires three. This last is always to be discouraged, but (ii) has this advantage over (i) that if $b \sin A>a$ we find it out from (ii) as soon as we have performed one addition and have looked up $\log a$, whereas if we use (i) we do not find it out till we have subtracted $\log a$ from $\log b$ and added $\log \sin A$.

The foregoing is intended to supplement Note 1177, not to amend it, and to supply a few hints which are certainly not in many of the textbooks (being indeed extraneous to an exposition of the theory) and which may be useful to some of the newer readers.

J. T. COMBRIDGE.

1225. *Addition to Note* 1209 (*Gazette*, vol. 20, p. 279).

Dr. Lidstone points out that the inequality

$$-n \log\left(1-\frac{t}{n}\right)-t \leqslant -\log\left(1-\frac{t^2}{n}\right), \quad 0 \leqslant t, \quad t^2<n, \quad n \geqslant 1$$

is deducible by term-by-term comparison if the expression on the left is written as

$$\frac{t^2}{2n}\left(1+\frac{2}{3}\frac{t}{n}\right)+\frac{t^4}{4n^2}\left(\frac{1}{n}+\frac{4}{5n}\frac{t}{n}\right)+\frac{t^6}{6n^3}\left(\frac{1}{n^2}+\frac{6}{7n^2}\frac{t}{n}\right)+\dots.$$

This argument gives us the inequality

$$e^{-t}-\left(1-\frac{t}{n}\right)^n \leqslant \frac{t^2}{n}e^{-t}-\left(1-\frac{t}{n}\right)^n(1-e^{-t^2/3n^2}).$$

Why do Whittaker and Watson speak of the inequality

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqslant \frac{t^2}{2n},$$

used by Bromwich, as " less precise " than that in which the function on the right is t^2e^{-t}/n ? Of the two functions, Bromwich's is the smaller if $t<\log 2$, theirs is the smaller if $t>\log 2$, and $\log 2$ is inside the range which t has to traverse. E. H. N.

1226. *The focus-sharing conics again.*

The quadrilateral property of common chords of three conics sharing three foci (*Gazette*, vol. 20, p. 182) is of course expressible in a projective form in which the circular points are replaced by any pair of points. It is not obvious, but it is true, that the circular points may be replaced by an undegenerate conic. Analytical verification is astonishingly simple. E. H. N.

1227. *The Representation of Numbers as the Sum of Cubes.*

It is well known that if n is a multiple of 6 it is possible to find integral values of x, y, and z satisfying the equation

$$x^3 + y^3 + 2z^3 = n. \quad \text{.....(i)}$$

In fact, if $n=6k$, we can take $x=k+1$, $y=k-1$, and $z=-k$.

Professor Mordell has suggested that perhaps this equation has integral solutions for all values of n (and indeed an infinity of such solutions). In support of this hypothesis, Mr. Chao Ko * has constructed a table showing solutions of the equation for all values of n up to 100, with the exception of $n=76$ and $n=99$. In the majority of cases the values of x, y, z are all small, but in some cases comparatively large values are required. Thus, when $n=57$, the simplest solution appears to be $x=96$, $y=259$, $z=-209$; and when $n=68$, $x=221$, $y=143$, $z=-190$. It may therefore be of interest to indicate a method which the author used for obtaining solutions in one or two of these more difficult cases.

Putting $2x=q-r$, $2y=-q-r$, $2z=p+r$, equation (i) becomes

$$3r(q^2 - p^2 - pr) = p^3 - 4n.$$

Hence we can replace it by the pair of equations

$$rs = \tfrac{1}{3}(p^3 - 4n), \quad \text{.....(ii)}$$

$$q^2 = p^2 + pr + s. \quad \text{.....(iii)}$$

We note that $p=x+y+2z$. Since $x^3 \equiv x \pmod 6$, it follows from (i) that $p \equiv n \pmod 6$. This restricts possible values of p and ensures that the right-hand side of (ii) is integral. To test whether a solution is possible for a given value of p we have therefore merely to factorize $\frac{1}{3}(p^3-4n)$ in all possible ways and see whether there is a pair of factors r, s which makes the expression on the right of (iii) a perfect square. If n is odd, so also is p and

* *Journal London Math. Soc.*, 11 (1936), 218-219.

therefore r, s, and finally q are all odd, so that we obtain integral values of x, y, z. If n is even, p is even, and therefore rs is divisible by 8. In this case the values of x, y, z are integral if we restrict the choice of factors so that r is even and s divisible by 4.

For example, when $n=68$, $p \equiv 68 \pmod 6$, and therefore p is of the form $6k+2$. Testing in succession the cases $k=0, \pm 1, \ldots$, we find that when $k=-3$, *i.e.* $p=-16$, $rs=-1456$; and if we take $r=-364$, $s=4$, $p^2+pr+s=6084=78^2$. Thus $q=78$, giving $x=\frac{1}{2}(78+364)=221$, $y=\frac{1}{2}(364-78)=143$, $z=-\frac{1}{2}(16+364)=-190$.

In the case $n=76$, for which a solution has not yet been found, it is possible to place further restrictions on the values of p to be tested. This depends on the fact that $x^3 \equiv 0, \pm 1 \pmod 9$ according as $x \equiv 0, \pm 1 \pmod 3$. Since here $n \equiv 4 \pmod 9$ equation (i) shows that x, y, z must all leave remainder $+1$ when divided by 3. If we replace p, q, r, s by $2p_1, 2q_1, 2r_1, 4s_1$, so that

$$r_1 s_1 = \tfrac{1}{8}(p_1^3 - 38), \quad q_1^2 = p_1^2 + p_1 r_1 + s_1,$$

we find that the restrictions on x, y, z imply the conditions

$$p_1 \equiv 2, \; q_1 \equiv 0, \; r_1 \equiv 2, \; s_1 \equiv 1 \pmod 3.$$

Hence $p_1 \equiv 2$, 5, or 8 (mod. 9), and similarly for r_1. But if $p_1 \equiv 2$, $r_1 s_1 = \frac{1}{8}(p_1^3 - 38) \equiv 8 \pmod 9$, and thus if $r_1 \equiv 2$, $s_1 \equiv 4$; if $r_1 \equiv 5$, $s_1 \equiv 7$; and if $r_1 \equiv 8$, $s_1 \equiv 1 \pmod 9$. In each of these cases,

$$p_1^2 + p_1 r_1 + s_1 \equiv 4 + 2r_1 + s_1 \equiv 3 \pmod 9,$$

while $q_1^2 \equiv 0 \pmod 9$. Thus we cannot have $p_1 \equiv 2$; and the case $p_1 \equiv 8$ can similarly be excluded.

Hence $p_1 \equiv 5 \pmod 9$; and therefore $p_1 \equiv 5$, 14, 23, or 32 (mod. 36). These possibilities can also be reduced by similar arguments. If $p_1 \equiv 5$, $r_1 s_1 \equiv 29 \pmod{36}$. Thus r_1, s_1 must be odd, and remembering $r_1 \equiv 2 \pmod 3$ we find that we have the possibilities

$$\begin{matrix} r_1 \equiv & 5, & 11, & 17, & 23, & 29, & 35; \\ s_1 \equiv & 13, & 19, & 25, & 31, & 1, & 7. \end{matrix} \pmod{36}$$

In each case $p_1^2 + p_1 r_1 + s_1 \equiv 25 + 5r_1 + s_1 \equiv 27 \pmod{36}$ and cannot be a perfect square. The case $p_1 \equiv 23$ can be excluded for similar reasons.

We are left with the possibilities $p_1 \equiv 14, 32 \pmod{36}$. Testing the smaller values of p_1 has not led to a solution of the original equation, nor has a solution yet been discovered for the case $n=99$.

W. HUNTER.

1228. *A group of coaxial spheres.*

The interesting property given recently in the *Gazette* (May, 1936, p. 140, Note 1185) by R. Goormaghtigh may be extended to space in the following manner:

Through the given point M, not in a face of the given tetrahedron $(T)=ABCD$, the three lines MUX, MVY, MWZ are drawn meeting

the three pairs of opposite edges DA, BC ; DB, CA ; DC, AB of (T) in the pairs of points, say, U, X ; V, Y ; W, Z ; let $U', \ldots, Z'$ be the traces on the same edges of the tetrahedral polar plane μ of M for (T).

The six spheres having their centres on the edges of (T) *and passing through the projections of the corresponding six pairs of points* U, U' ; ... ; Z, Z' *upon a given plane form a coaxial net.*

The points U, U' are harmonically separated by the points D, A, hence the projections L, L' of U, U' upon the given plane Σ are harmonically separated by the projections D', A' of D, A upon Σ. Consequently the sphere (U) having its centre on the edge DA and passing through L, L' is orthogonal to the sphere $(D'A')$ having $D'A'$ for diameter. Now the spheres (D), (A) having the points D, A for centres and DD', AA' for radii are also orthogonal to the sphere $(D'A')$, hence the three spheres (D), (A), (U) are coaxial. Thus any sphere orthogonal to (D) and (A) is also orthogonal to (U), and in particular (U) is orthogonal to the orthogonal sphere (R) of the spheres (D), (A) and their two analogues (B), (C).

The centre U_0 of the sphere (U) is the midpoint of the segment UU', hence to prove the announced proposition it is sufficient to show that the point U_0 and its five analogues $V_0, \ldots, Z_0$ are coplanar, which I shall proceed to prove.

With the points D, A, B, C as centres, four spheres (P), (Q), (R), (S) may be described so that one of their planes of similitude will coincide with the plane μ.* The pairs of points U, U' ; ... ; Z, Z' will be the centres of similitude of the spheres (P), (Q), (R), (S) taken two-by-two, and the midpoints $U_0, \ldots, Z_0$ of the segments $UU', \ldots, ZZ'$ will be the centres of the six spheres of similitude of these spheres taken two-by-two, hence these six points are coplanar.

Special cases of the proposition may be obtained by making the point M coincide with the centroid of (T), or with its incentre, etc. The reader may find it interesting to formulate these propositions.

NATHAN ALTSHILLER-COURT.

1229. *Queer Alternatives.*

§ 1. There are *two* essentially different alternative situations under which

(α) A triangle PQR can be inscribed in another triangle ABC, with P, Q, R respectively on BC, CA, AB, such that the orthocentre of the inscribed triangle is the circumcentre of the outer one : viz.

(1) when the triangle PQR is similar to the triangle ABC ; in this case the circles AQR, BRP, CPQ pass through the circumcentre of ABC ;

(2) when the triangle PQR is self-polar to the circumcircle of the triangle ABC and, in this case, the triangles are in perspective with the centre of perspective on the circumcircle of ABC.

* Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 206. The Macmillan Company, 1935.

Is it possible to bridge the gulf between these queer alternatives and discover a rational link to connect them ?

Further, there seems to be no simple geometrical proof available to show that these are the only possible cases for (α) to happen.

§ 2. It is interesting to add the reciprocal theorem.

(β) A triangle PQR can be inscribed in another ABC, with P, Q, R on BC, CA, AB respectively, such that the incentre or an excentre of PQR coincides with the orthocentre of ABC only when either

(1) the triangle PQR is similar to the pedal triangle of ABC, in which case the circles AQR, BRP, CPQ pass through the orthocentre of ABC ; or

(2) the triangle ABC is self-polar to the incircle or an excircle of PQR, in which case, again, the triangles are in perspective, with the axis of perspective as a tangent to the incircle or an excircle of PQR.

§ 3. The problem of finding Q, R on AC, AB respectively, given P on the side BC of a triangle ABC, so as to satisfy the condition (α) or (β), is easy in view of the properties enunciated above ; but if P lies elsewhere, it is difficult to fix the triangle PQR so that it may have a given point as its orthocentre, incentre or excentre.

Will any geometer interest himself in these problems ?

A. A. KRISHNASWAMI AYYANGAR.

1230. *Euclid XI*, 4.

Note 1180 by Mr. W. J. Dobbs gives two proofs of Euclid XI, 4 ; the first of them is a well-known proof attributed to Legendre (vide *The Thirteen Books of Euclid's Elements* by Sir Thomas L. Heath, 2nd edition, p. 280), while the second occurs on page 9 of *The Elements of Synthetic Solid Geometry* by N. F. Dupins, 1893. I may be pardoned for suggesting a third proof, of which the above can be regarded as special cases.

Let OC be perpendicular to OA, OB, and OX any other line through O in the plane AOB.

Through X draw *any* line AXB. Then, by Euler's theorem,

$$XB \cdot AC^2 + AX \cdot BC^2 = AB(CX^2 + AX \cdot XB),$$

$$XB \cdot AO^2 + AX \cdot BO^2 = AB(OX^2 + AX \cdot XB).$$

Hence

$$XB(AC^2 - AO^2) + AX(BC^2 - BO^2) = AB(CX^2 - OX^2),$$

$$i.e. \quad XB \cdot OC^2 + AX \cdot OC^2 = AB(CX^2 - OX^2),$$

$$i.e. \quad AB \cdot OC^2 = AB(CX^2 - OX^2).$$

Therefore $$OC^2 = CX^2 - OX^2.$$

So, COX is a right angle.

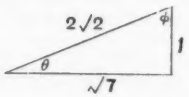
A. A. KRISHNASWAMI AYYANGAR.

1231. *Trigonometrical Note.*

Have I thought of a *new* question in elementary trigonometry? Impossible! All the same it may be new to some teachers and makes a good foundation for an hour's work.

Given a table of logarithmic sines, calculate a table of logarithms.

It is easy to get log 2 and log 3. The fun begins with log 7.



Take the triangle shown; $\log \sin \theta$ is found from log 2. Hence θ from the table; hence its complement ϕ; hence $\log \sin \phi$ and log 7.

Oddly enough the above triangle gives a much more accurate value of log 7 than the triangle with sides 25, 24, 7, which would serve equally well theoretically. Perhaps someone will supply the reason.

At any rate the above method, or obvious variations of it, can be carried as far as may be desired. C. O. TUCKEY.

1232. *On Note* 1187.

With reference to Note 1187 in the May *Gazette* (1936)

if $$a^3+b^3=c^3+d^3,$$

then: I $$(a+\lambda_1)^3+(b+\lambda_1)^3=(c+\lambda_1)^3+(d+\lambda_1)^3,$$

$$\text{provided} \quad \lambda_1=-\frac{a^2+b^2-c^2-d^2}{a+b-c-d}.$$

II $$(a+\lambda_2)^3+(b-\lambda_2)^3=(c+\lambda_2)^3+(d-\lambda_2)^3,$$

$$\text{provided} \quad \lambda_2=-\frac{a^2+d^2-b^2-c^2}{a+b-c-d}.$$

III $$(a+\lambda_3)^3+(b-\lambda_3)^3=(c-\lambda_3)^3+(d+\lambda_3)^3,$$

$$\text{provided} \quad \lambda_3=-\frac{a^2+c^2-b^2-d^2}{a+b-c-d}.$$

So, starting with

$$3^3+4^3=(-5)^3+6^3,$$

$$\lambda_1=6, \quad \lambda_2=-\tfrac{2}{3}, \quad \lambda_3=3$$

and the corresponding results are

$$9^3+10^3=1^3+12^3, \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$7^3+14^3=(-17)^3+20^3, \quad \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

$$6^3+1^3=(-8)^3+9^3. \quad \ldots\ldots\ldots\ldots\ldots\ldots (c)$$

Each of these will give rise to 2 new results (not 3) and the process can be continued indefinitely. E. P. LEWIS.

REVIEWS.

The theory of the properties of metals and alloys. By N. F. MOTT and H. JONES. Pp. xiii, 326. 25s. 1936. International series of monographs on physics. (Oxford)

This book is an application of the most advanced and most abstract methods of mathematical physics to a subject of highly practical importance, the science of metals. Even if the technical metallurgist should not be able to read this work, it will influence technical progress indirectly by its brilliant representation of the results of physical research. Professor Mott, to whom we owe already another important book (*The theory of atomic collisions*, written together with Professor Massey), and his collaborator Dr. Jones deserve great credit for collecting the results of innumerable papers scattered over all the scientific periodicals in a handy volume. This book is admirable in every respect: by the choice of the material, the clearness of the style, the precision of the mathematics, the simplicity of the figures and the completeness of the tables. It seems to be the first complete presentation of this subject in English, though there are several in German (the big article by Sommerfeld and Bethe in vol. XXIV, 2 of the *Handbuch d. Physik*, 2nd edition, and a new book by H. Fröhlich).

Out of the rich content of the book I can mention only a few major points. The introductory chapter treats the thermal properties of crystal lattices; especially valuable is the account of the new ideas of Bragg on "order and disorder". The following two chapters deal with the equilibrium of the electrons in the lattice and their motion under the influence of an external field, including the optical and X-ray phenomena. The next chapter contains the theory of cohesion forces which determine the elastic properties of the metals and the binding energies of the electrons and ions in the substance. Then there follows an extensive discussion of the different crystal structures, chiefly with respect to the position of the discontinuities of energy in the phase space (Brillouin's zones). The last two chapters deal with the caloric and magnetic properties of metals, and with electric and thermal conductivity in pure metals and alloys. At the end of the volume there are several appendices which contain amongst other things a valuable collection of the most important constants of metals, and a table of universal constants and conversion factors used in the book.

This summary gives only a very restricted impression of the great variety of problems and results. I can strongly recommend the book to everybody who wishes to become acquainted with the modern theory of metals.

MAX BORN.

Differential Calculus. By T. CHAUNDY. Pp. xiv, 459. 35s. 1935. (Oxford University Press)

This is a very difficult book to review fairly. It is very individual in its choice of subject-matter, and although in some topics it is very exhaustive—sometimes too much so—and although it includes a certain amount on differential equations and on analytic continuation (including questions of branch points), it deliberately turns its back on even the simplest study of integration (except in one spot near the end) and on the complex variable. In this respect one is reminded of the old study of geometrical conics in which analytic methods were disdained.

The book is rigorous and, indeed, many of the earlier pages and earlier examples—both those worked out and those set for the reader—would be difficult to appreciate without a preliminary reading of the major part of such

a book as Knopp. Some examples are so difficult in themselves that they obscure the point which is to be illustrated. Nevertheless, in spite of the rigour, some of the definitions and conventions are most perversely calculated to confuse the reader, and this in places where the traditional procedure is much more straightforward.

For example, let us consider the introduction of the hero of the drama. The derivative is not defined in one step. On one page is the definition of differentiability, viz. "A function $f(x)$ such that the ratio $[f(x)-f(\xi)]/(x-\xi)$ converges as $x \to \xi \pm$ is said to be differentiable at $x=\xi$", while the definition of the resulting derivative appears on the next page. To understand the former definition we go back and find first—as a minor obstacle—that Chaundy does not define convergence apart from convergence to a particular limit A. We also find that "$f(x) \to A$ as $x \to \xi$ or, for special emphasis, $x \to \xi \pm$ if $f(x)$ tends to A whether from below or from above." We notice that the point ξ itself is definitely excluded—as, of course, it must be in the case of the incremental ratio—and on page 28 Chaundy insists on the difference between a function which "converges at a point", *i.e.* which has a common limit on both sides, and a function which is "continuous at the point". Nevertheless, a theorem is stated that "if $f(x) \to A$ as $x \to \xi$ and if $x_n \to \xi$ then $f(x_n) \to A$". This theorem can be seen to be false by taking $\xi=0$, $f(x)=1 (x \neq 0)$, $f(0)=0$, $x_n = \frac{1}{n} \sin n\pi/2$. The theorem ought to be true and would be true if the usual meaning had been assigned to "$f(x) \to A$", but then the definition of differentiability should have been different. It is all very apt to mislead the student where he is only too prone to go astray.

There is a somewhat similar convention that when a function is not everywhere defined "continuity" means "continuity excluding points of non-definition". In particular, if a series converges for $x \leqslant 1$ but does not converge for $x > 1$, following the convention the sum-function is said to be continuous at $x=1$ when other writers would say it is continuous on the left at $x=1$. Thus Theorems 27 and 28 assert: "The sum-function of a power series is continuous wherever the power series converges". We can easily imagine a student arguing as follows:

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \ldots \quad (0 \leqslant x \leqslant 1);$$

hence $\log(1+x)$ is continuous at $x=1$. Now the student's statement happens to be true, but his proof is faulty. It is not $\log(1+x)$, but the sum of the series which is proved to be continuous. The only correct inference about $\log(1+x)$ is that it is continuous to the left at $x=1$. It is because students are easily muddled in a rarefied region of thought where they find it difficult to breathe at all that teachers know they must be pedantically clear in their statements. It takes longer to write out that the sum-function is continuous at every point within its interval of convergence and is continuous towards the interval at either end at which it converges, but the additional effort is not wasted. It emphasises the distinction between two cases of which one is much more elementary than the other.

Another point deserves to be mentioned here and, if possible, widely advertised. So far as the reviewer knows even the best English and many good Continental treatises on the Calculus fall into a trap in proving the formula for the derivative of a function of a function. The trap is made more plausible by the conventions used in this book. Suppose y is a differentiable function of u, u a differentiable function of x. In the proof it is forgotten that δu must not be zero infinitely often if $\lim \delta y/\delta u$ is to have a meaning. The case in which $du/dx \neq 0$ can be taken in the usual manner, but that in which $du/dx = 0$

requires some delicacy of handling—see, for example, Kowalewski, *Calculus*, 3rd ed., p. 61. An elementary work might well omit this case, but a thorough treatise should be more careful.

Let us now turn to less important matters. Firstly, the reviewer welcomes a suggested notation $(n\,!\,r)$ for $\binom{n}{r}$. No doubt objections can be raised, but the question of the most suitable notation is worthy of investigation by a joint committee of mathematicians and printers.

On the other hand, have Oxford Groups found their way into mathematics? The book calls a set of points a "group". It calls a closed system a "complete system", and by two definitions it concludes that $0^0=1$ and that $0^0=0$, instead of concluding that 0^0 is meaningless. It is stated that the only type of discontinuity of an algebraic function is a pole. By a slip the word "rational" has been omitted (example, $\operatorname{sgn} x=\sqrt{(x^2)}/x$ is surely algebraic). By the way, $\log|x|$ is said to have a "pole" at $x=0$.

It is said to be "best to consider the origin a point of infinite oscillation of such a function as $\exp(1/x)+\sin(1/x)$". Here classification has gone too far. It is better to lump together all discontinuities which are not "simple".

There is an omission in considering the derivative of $\log x$. It is necessary to consider the limit of $(1+x^{-1})^x$ not only as $x\to+\infty$, but also as $x\to-\infty$.

When it is said that the positive signs must be taken for the derivatives of the inverse trigonometric functions when the principal values of these functions are taken, it is not clear that this only applies to $\sin^{-1}x$, etc., and not to $\cos^{-1}x$, etc.

One criticism, though perhaps of minor importance, deserves fuller treatment. On page 142 an example is given showing that the condition that $f(x, y)$ is differentiable along every curve through a point is not sufficient to ensure that it is differentiable according to (8), that is,

$$\delta f=A(\delta x,\delta y)\delta x+B(\delta x,\delta y)\delta y,$$

where A, B converge as δx, $\delta y\to 0$. (It is to be understood that $\dot{x}$, $\dot{y}$ exist at t_0 on the curve.) The example given is $f(x, y)=|xy|^{\frac{1}{2}}$, and the reasoning breaks down because if

$$\delta x=A\delta t,\ \delta y=B\delta t \text{ then } \delta f=|AB|^{\frac{1}{2}}|\delta t|, \text{ not } |AB|^{\frac{1}{2}}\delta t.$$

The function is not differentiable along every curve through the origin. The example should have been $f=x^{\frac{1}{3}}y^{\frac{2}{3}}$. However, the suggested relation (to be proved untrue) is only made plausible by the conditions of differentiability being divorced from the value of the derivative. The expected theorem is that condition (8) will be satisfied if constants A_0, B_0 exist such that $\dot{f}$ exists and is equal to $A_0\dot{x}+B_0\dot{y}$ along any suitable curve. This is true, though the proof is not altogether elementary.

There was here a great opportunity for a rigorous book in English on the calculus with ample room for logical development and catering for the kind of reader who does not expect to be amused by long formulae or applications. The author does not pander to those who want calculus easy or picturesque. And yet, somehow, the book does not satisfy. It does not depend on references to other books for its fundamental proofs, and confine itself to more advanced aspects. Nor, on the other hand, does it include sufficient about convergence, uniform continuity, integration, etc., to make a complete whole.

An outline of the ground covered is added. After an introduction, through continuous functions and derivatives, Taylor's series, partial derivatives and indeterminate forms are considered and make up a traditional though rather

expanded course. After a good chapter on analytic functions the book returns to maxima and minima. Finally there are extended treatments of implicit functions, Jacobians and differential operators. As a kind of appendix there is a last chapter on hypergeometric and trigonometric expansions.

It is clear that the book would be very useful to students who had a supervisor at hand to clear up difficulties. P. J. D.

Introduction to the Theory of Linear Differential Equations. By E. G. C. POOLE. Pp. viii, 202. 17s. 6d. 1936. (Oxford)

This book helps to fill a conspicuous gap in the English mathematical bookshelf. Craig's *Treatise on Linear Differential Equations* (New York, 1889), which did not proceed beyond its first volume, and the fourth volume of Forsyth's comprehensive *Theory* (1902) represent the points of view of a bygone generation, and indicate paths of approach now seldom trodden. Although those works are of enormous historical interest, and in fact are indispensable to the specialist, they do not meet the needs of those students of to-day who are attracted to linear differential equations, either to study them for their own interest or to use them in problems of applied mathematics. Thus there really was a need for a book that selected from the body of older theory some of the matter that still remains of value, to supplement it with typical and specially interesting samples of more modern work, but particularly with the more interesting and useful results of investigations on the linear equations that spring out of problems in mathematical physics. Here the choice is purely a matter of personal taste, guided by experience, yet so great is the scope that to choose to the readers' greatest advantage is no easy task. The author has chosen well.

The first five chapters of Poole's *Linear Differential Equations* contain a very lucid treatment of the principles of the subject, both in the real domain and in the domain of the complex variable, as established during the last century, together with certain natural developments, based on these principles that have taken place in recent years. The classical methods which to the present time have formed the backbone of any serious textbook on differential equations can be accepted as essential or inevitable only so long as they maintain an indisputable superiority over other methods of approach, and it does seem that ideas are now in embryo that in due course will develop into processes to which the classical methods will have to give place. A hint of this tendency appears in Poole's second chapter, which shows the very fruitful results of applying the Heaviside operational calculus to linear equations with constant coefficients. This example, however, is a minor development compared with the very significant advance made during the last ten years on the basis of the ideas of matrix theory, an advance to which, unfortunately, no allusion is here made.

The last five chapters deal with those equations of the second order which are interesting, both because of the fair degree of completeness to which the theory of their solutions has reached and on account of the frequency of their occurrence in mathematical physics, namely, the hypergeometric equation with its particular and confluent cases and the Lamé and Mathieu equations. In these chapters the author displays a skill in exposition that can come only from a deep interest in the functions defined by these equations and a very intimate knowledge of the work that has been done on them. This account should serve to draw more workers into this field, in which much still waits to be discovered. The exercises on these, as well as on the earlier chapters, are excellent material on which skill may be tested and technique developed.

Personally, the present reviewer would have been more pleased with the

book had it contained a considerably fuller account of the Sturm-Liouville theory of oscillations and its consequences, but no doubt limitations of space decided the author against its inclusion. The one serious complaint that can be made is that the publishers have chosen to price the book so dearly. The author is to be thanked for the excellent text and admirable examples, and it only remains to wish the book the success that it deserves. E. L. I.

Lehrbuch der darstellenden Geometrie. By the late EMIL MÜLLER. Fourth Edition revised by ERWIN KRUPPA. Pp. viii, 390 with 366 figures. In one vol. or in parts: I. Projektion auf eine Bildebene; II. Zugeordnete Normalrisse, Krumme Flächen; III. Axonometrie, Perspektive, Landkartenwürfe. Foreign prices: in one vol., RM. 16.50; in parts, RM. 5.85, 5.10, 6.45. 1936. (Teubner)

Because mathematical teaching in Great Britain is now more than ever subject to the domination of one school, which, exalted though it is in the realm of pure theory, affects not to know of the needs of the technician, it is unlikely that descriptive geometry will ever find its place in the curriculum of our schools and colleges. Students and teachers from the technical schools of the Continent have again and again expressed to the present reviewer their amazement that our future engineers should manage to muddle along with no knowledge of descriptive geometry, or at most with that untheoretical knowledge of special dodges acquired incidentally in the drawing office.

That being so, it is difficult to decide how to introduce a book such as that under review to a public taught to believe that no true mathematician should recognise any branch of mathematics that is likely to be useful to any one else. One can only go on the assumption that among that public there may be one or two heretics (let us say) who so far depart from orthodoxy as to be prepared to examine the hypothesis that in descriptive geometry is to be found a fascinating branch of pure and a fruitful branch of applied mathematics (for why should those who profess mechanics monopolise that designation ?). To such one might propose, in the first place, one of the smaller books of the French school, for in them descriptive geometry can be seen in a pure—almost puritan—form, without the distraction of too strong an appeal to analytical geometry on the one hand, or of too emphatic an insistence on the practical applications on the other. When the essence of the subject has thus been absorbed, it is time to examine the contacts with more theoretical geometry and to see best how to apply the knowledge acquired to practice. Then a book of the German type is to be recommended, and there can be none more suited to the purpose than that now under review.

The original *Lehrbuch der darstellenden Geometrie* by the late Dr. Emil Müller, formerly professor in the Technische Hochschule in Vienna, was published in two volumes, of 368 and 361 pages, in 1908 and 1916 respectively. The latest edition has been prepared by Dr. Erwin Kruppa, of the same institution, and that with supreme skill, for in spite of the drastic reduction in the size of the book, nothing of real importance has been jettisoned; on the contrary, new matter (for example, an excellent section on photogrammetry) has been introduced. The book is one that cannot be praised too highly; every process is founded on a rigorous theoretical basis, the whole subject is developed in an easy and logical fashion, and every practical application springs naturally from out of the main stream of reasoning. In fact, it brings out clearly the twin features of descriptive geometry, the essential and the accidental. For in essence descriptive is a branch of pure geometry, and always will be so, but by the accident that no practical device for making solid drawings has yet been invented, it is a tool in the hands of the intelligent

technician which will become obsolete when, but only when, three-dimensional draughtsmanship has become as practicable as that upon the flatness of the drawing board. The scope of Müller-Kruppa's applications is wide—architecture, machine construction, road and bridge building, map making contribute their share, both illustrating the principles and enhancing the value of the book as a practical manual. The numerous illustrations are excellent, clear, of sufficient detail, and yet restrained; the typography is very good indeed. The copy sent for review is in three parts, bound in stout but flexible covers, a most convenient arrangement. E. L. I.

Theorie der ebenen Elastizität unter benutzung eines Systems hyperkomplexer Zahlen. By L. SOBRERO. Pp. 50. RM. 3. 1934. Hamburger Mathematische Einzelschriften, 15. (Teubner)

This is one of a series of mathematical monographs and deals primarily with a kind of generalised theory of functions, standing to the biharmonic equation in the same relation that the usual functions of a complex variable stand to Laplace's equation.

The basis of the procedure is simple; precisely as one can introduce a symbol i, about which nothing need be known save that it satisfies the equation $1+i^2=0$, so that all higher powers of i occurring in an algebraic expression are reducible to i^0 and i^1, so we can introduce a symbol j, satisfying the equation

$$1+2j^3+j^4=0,$$

and, by means of this, any algebraic expression involving j can be expressed in the form

$$a+jb+j^2c+j^3d,$$

j, j^2, j^3 being treated as irreducible. The four parts of any function of $x+jy$ then satisfy separately the biharmonic equation.

For the complete treatment a four-dimensional system of variables is required, but an important special case occurs when only two variables are present, and it then appears that the functions a, c, d can be identified with the stresses $\widehat{yy}$, $-\widehat{xx}$ and $\widehat{xy}$ in two-dimensional elastic theory.

The greater part of the monograph is devoted to the theory of the above "hypercomplex" numbers, including the condition for an "analytic" function and the establishment of certain important analogues with ordinary complex theory, *e.g.* Cauchy's and Taylor's theorems and the method of residues.

In the last twenty pages the author applies his theory to the problem of plane strain or plane stress, and gives the solution of a few problems as examples. None of the results are new in themselves, but their deduction in this manner is of considerable mathematical interest.

The title seems hardly appropriate, for this is not the sort of work which is suitable for a student wishing to learn about plane Elasticity. One has a feeling that the Elasticity is there to illustrate the Pure Mathematics, and only because it happens to be a branch of Mathematical Physics where the biharmonic equation is fundamental.

On the other hand, to one already familiar with the theory of Elasticity and anxious to extend the range of his mathematical tools, this little booklet can be heartily recommended and will be found most stimulating reading.

It suffers occasionally from over-condensation; thus the evaluation of the integral $\int_{-\infty}^{\infty} \{(1+jx)/(1+x^2)^2(z-x)\}\,dx$ on p. 27 appears to require a good deal more than "a few easy simplifications", and the expressions for the displacements, merely quoted on p. 32, were worth proving with some care, more

particularly as nothing is said of the arbitrary rigid body rotation which ought to be present.

One could wish, also, that the author had investigated the relation between his hypercomplex functions and the usual stress function.

On the whole, however, the monograph is very clear and readable, and bears the mark of an expert hand. It is to be hoped that all interested in biharmonic theory will study it. L. N. G. F.

Einführung in die Theorie der Systeme von Differentialgleichungen. By E. Kähler. Pp. iv, 79. Foreign price: RM. 3; bound, RM. 3.75. 1934. Hamburger Mathematische Einzelschriften, 16. (Teubner)

No. 16 of the Hamburg series of monographs deals with developments of the Cartan theory of Pfaffian systems. The introductory pages sketch briefly the calculus of differential forms, including the concept of differential ideals. Then a discussion which brings in the fundamental notions of integral manifolds, integral elements, regular chains of integral elements and their characteristic numbers leads up to two fundamental theorems of existence and uniqueness which are the nucleus of the theory. But existence theorems (to some of us at least) lack conviction unless supported by practicable devices for manipulation, and so there follow methods for determining the regular chains and characteristic numbers belonging to a set of Pfaffian forms, for establishing criteria for regular chains and the like. Definite applications of the theory are made to total differentials, completely integrable Pfaffian systems, differential equations for the characteristic surfaces in space of two complex variables, and to systems of partial differential equations. Contact between this theory and the Lie group theory is established in an appendix. It is surprising to find how much useful material there is in this little book.

E. L. I.

Wahrscheinlichkeitsrechnung und allegemeine Integrationstheorie. By E. Tornier. Pp. vi, 160. RM. 9. 1936. (Teubner)

Let no one buy this book hoping to obtain an account of Probability as it is usually understood. More than half the book is devoted to an abstract theory of the measure of sets of a general character and of the corresponding integrals. This is done without any reference to anyone, not even Fréchet, de la Vallée Poussin or Carathéodory. The style is forbidding, and there is no attempt to help the reader to see what is intended. The only derivative considered is essentially a net-derivative, and it is pointed out that any *two* such net-derivatives are equal nearly everywhere, but there are a more than countable infinity of possible nets, and the measure of the sum of the sets where the net-derivatives are not equal to a particular one may be positive. In the case of n-dimensional Euclidean differentiation this difficulty is overcome by Vitali's theorem or by such a masterly method as that used by de la Vallée Poussin. Professor Tornier does not discuss it even so far as to point out the difficulty of bringing in the essential concept of " shape " in abstract spaces.

On page 101 we come to a discussion of probability proper, but Professor Tornier has apparently never heard of J. M. Keynes or other critics of the fundamental notions. He writes glibly of obtaining probabilities in some cases " näherungsweise ", though he does not say how this can be done without a circular use of Bayes' formula or else by the often disproved limit theories, such as Venn's. Professor Tornier merely assumes that the probability is a number satisfying the postulates for a general mass. With this he is able to obtain the usual general theorems on probability, including Cheby-

shef's. (Why, by the way, do we spell Russian names as if they were first turned into German ?) Within 25 pages of the end the Gauss error function occurs, but only in connection with the more searching asymptotic approximations for total probabilities (Khinchine).

There seems little to recommend the book to a British audience. It is astonishing, however, that a book of so abstract a character should be accepted in modern Germany, where, according to publications received by many of us, mathematics is to be made concrete and "nordic". P. J. D.

Eindeutige analytische Funktionen. By R. NEVANLINNA. Pp. viii, 353. Geh. RM. 27.60; geb. RM. 29.40. 1936. Grundlehren der mathematischen Wissenschaften, 46. (Springer)

Professor Nevanlinna's talent for unification is well known. Like an expert solver of jig-saw puzzles, he can fit together the most jagged of theorems to form a beautiful and unsuspected pattern, and he can supply any pieces that are missing. Here he deals with some properties of conformal representation. Perhaps the result is not quite so striking as his work on meromorphic functions, or is it just that the subject cannot be grasped very easily ? At any rate, one has the feeling that he has said the first and the last word on a number of topics.

Unfortunately it is not so easy to explain what these topics are, nor even what they are about. The concept of harmonic measure, which dominates much of the book, is anything but simple. It arises somewhat as follows. Suppose that $f(z)$ is regular in a domain G and that $|f(z)| \leqslant M$ in G. Suppose, moreover, that $|f(z)| \leqslant m < M$ on a part α of the boundary of G. How can this fact be used to improve the previous inequality ? It can be and has been used in a variety of cases with striking success. Our aim is to unify these results.

Consider first the case when G is the unit circle G_0 and α an arc of the circumference Γ. Poisson's formula is

$$f(re^{i\phi}) = \int_0^{2\pi} \frac{f(e^{i\theta})\, d\theta}{1 + r^2 - 2r\cos(\theta - \phi)},$$

so that

$$|f(re^{i\phi})| \leqslant m \int_\alpha \frac{d\theta}{1 + r^2 - 2r\cos(\theta - \phi)} + M \int_{\Gamma - \alpha} \frac{d\theta}{1 + r^2 - 2r\cos(\theta - \phi)}.$$

This suggests that we should define the harmonic measure of the arc α at the point $re^{i\phi}$ in regard to G_0 to be

$$\omega(re^{i\phi};\ \alpha, G_0) = \int_\alpha \frac{d\theta}{1 + r^2 - 2r\cos(\theta - \phi)},$$

so that $$|f(z)| \leqslant m\omega(z;\ \alpha, G_0) + M\omega(z;\ \Gamma - \alpha, G_0).$$

For fixed α and G_0, ω is a harmonic function of z which takes the limiting value 1 as z approaches α and the value 0 as z approaches $\beta = \Gamma - \alpha$.

Now all this can be generalised to any connected domain G. If α is part of the boundary of G, and β the remainder, there is a unique function harmonic in G which takes the value 1 on α and the value 0 on β. This is defined to be $\omega(z;\ \alpha, G)$. It is closely connected with the Green's function of the domain. The utility of the conception depends on the fact that ω behaves in a simple way if G is enlarged or transformed. Suppose that we make G bigger by making parts of β bulge outwards, keeping α fixed. Then, for fixed z, ω becomes larger. This is *Carleman's principle of extension of domains.* Or,

again, suppose that $w(z)$ is regular and uniform in G_z and that its values form a domain G_w. Let α_z correspond to α_w. Some or all of α_w may enclose a part A_w of G_w. Let G_w, less this, be G_w^*. Then

$$\omega(z\,;\ \alpha_z, G_z) \leqslant \omega\{w(z)\,;\ \alpha_w, G_w^*\}.$$

This is *the principle of harmonic measure*, from which numerous known results, for example, Hadamard's three circles theorem and Phragmén-Lindelöf results, can be easily deduced.

Next we come to Ahlfors' fundamental distortion theorem, by which he proved (as Denjoy had conjectured more than twenty years before) that an integral function of order k cannot have more than $2k$ asymptotic paths. Here Professor Nevanlinna has missed an opportunity. For he makes no mention of what is perhaps the most important application of Ahlfors' theorem, the inequality for a function which takes no value more than p times in the unit circle which Miss Cartwright published last year; and a reference to Macintyre's paper on the Denjoy-Ahlfors theorem and the related work of Grötzsch would not have come amiss.

There are five chapters on meromorphic functions—rather similar to the author's Borel tract, but with new matter interpreting the results in terms of non-euclidean geometry. Other chapters are about sets of harmonic measure zero (there is no space to explain what this is), singularities and the associated Riemann surfaces, and Ahlfors' recent work on overlapping surfaces. Altogether this is a most important book. J. M. W.

Osiris : Studies on the History and Philosophy of Science, and on the History of Learning and Culture. I. The David Eugene Smith Presentation Volume. Edited by GEORGE SARTON. Pp. 777. 1936. (The Saint Catherine Press, Bruges)

The above title and sub-title on the outside cover of this portly volume amply describe its nature and purpose. It is a volume of studies on the history of mathematics and on the history of science by divers authors and in divers tongues, drawn up and edited to celebrate the seventy-sixth birthday of Professor David Eugene Smith, the eminent mathematical historian. Mathematicians have in the past been more than neglectful of the historical aspect of their chosen subject, but that mathematician would indeed be hard to please if he did not find something to interest and something to instruct in this publication.

The following gives the table of contents with their respective authors: "Bibliography of the historical writings of David Eugene Smith" (Bertha M. Frick); "Unpublished letters of J. J. Sylvester and other new information concerning his life and work" (R. C. Archibald); "The editing of early learned journals" (S. B. Barnes); "Certain unpublished letters from French scientists of the revolutionary period taken from the files of Joseph Lakanal" (Charlotte H. Boatner); "L'algebra nella storia e nella preistoria della scienza" (E. Bortolotti); "Origins of analytic geometry" (J. L. Coolidge); "The mysterious Abolays" (G. O. S. Darby); "A note on a tenth-century graph" (H. G. Funkhouser); "The sources of al-Khowarizmi's algebra" (S. Gandz); "Samuel Morey, a pioneer of science in America" (F. H. Getman); "The scientific value of the Copernican induction" (B. Ginzburg); "Legendre and the French reform of weights and measures" (C. Doris Hellman); "Samuel Lee (1625-91), a clerical channel for the flow of new ideas to XVIIth century New England" (T. Hornberger); "Caesar's astronomy by Peter Apian, Ingolstadt 1540" (S. A. Ionides); "The influence of Thomas Digges on the progress of modern astronomy in XVIth

century England" (F. R. Johnson); "The first printed arithmetic of Spain. Francesch Sanct Climent: *Suma de la art de arismetrica*, Barcelona, 1482" (L. C. Karpinski); "Michel Chasles e la teoria delle sezzione coniche" (G. Loria); "Una obra astronomica desconocida de Johannes Avendaut Hispanus" (J. Millas i Vallicrosa); "The number e" (U. G. Mitvhell and Mary Strain); "Une lettre inédite de Newton à Pepys" (J. Pelseneer); "Three unpublished calendars from Asyut" (A. Pogo); "François Legendre, arithméticien" (Vera Sanford); "Montucla (1727-99)" (G. Sarton); "Les Mathématiques dans le *Journal des Savants*. Première période, 1666-1701" (P. Sergescu); "Short stories in colonial geometry" (L. G. Simons); "On the use of series in Hindu mathematics" (A. N. Singh); "Coelestinus' summary of Nicolas Oresme on marvels. A fifteenth-century work printed in the sixteenth century" (L. Thorndike) "Die Siebeneckabhandlung des Archimedes" (J. Tropfke); "The Indian prelude to European mathematics" (D. Uvanovic); "Les plus anciennes traductions latines médiévales (X^e et XI^e siècles) de traités d'astronomie et d'astrologie" (A. Van de Vyver); "Quatre notes sur les mathématiques babyloniennes" (Q. Vetter); "Bemerkungen zu den quadratischen Gleichungen der babylonischen Mathematik" (K. Vogel); "Snellius' notes on the reflection and refraction of rays" (J. A. Vollgraff); "An unpublished hydraulic experiment of Roberval 1668" (Helen M. Walker); "The tradition behind Galileo's methodology" (P. P. Wiener); "Die Tafeln von Toledo" (E. Zinner).

The reviewer has no option but to select articles at random. The individual reader must make his own choice. For those who are interested in genuine mathematical history, the section on the number e is excellent. It develops the relationship of π and e from earliest times, until we obtain Euler's masterstroke $e^{\pi i} = -1$. It shows that e bears much the same relationship to the rectangular hyperbola that π does to the circle and ellipse. The original method of obtaining various expansions connected with the exponential and the logarithm are set forth. The proof of the transcendental character of e is also dealt with, and the appropriate rôles played by such men as Napier, Briggs, Oughtred, Lindemann, to name only a few, are duly explained. For the reader who loves rather the purely personal history of the great mathematicians, there is provided further information about J. J. Sylvester, who, in addition to his mathematical achievements, coached Florence Nightingale in the rudiments of mathematics. It is well known that he had strong views on the rules underlying poetry and that he himself took enormous pains with his own poetical compositions. It is, however, not so well known that his friends were also being constantly called in to give aid, and that he thought that the only way to obtain a tolerable translation of a few hundred lines of Horace "would be to parcel the work out among the members of a society who should each devote the labour of a year to effect the adequate rendering ... the results to be submitted to the criticism and correction of the joint body to be periodically convened for the purpose". In his paper on Newton's rule for the discovery of imaginary roots (1864), he quotes from the fifth act of Shakespeare's *Midsummer Night's Dream*:

> "Turns them to shapes and gives to airy nothing
> A local habitation and a name."

On the other hand, students of social philosophy, involving such questions as security of tenure for all grades of workers, will be interested to hear that "Roberval's secretiveness with respect to his discoveries and his reticence about publishing is generally attributed to the fact that from 1634 until his death in 1675 he was obliged, at intervals of three years, to defend, by open competition, his occupancy of the chair of Ramus in the Collège Royal; in

this mathematical tournament, imposed by the founder of the chair, questions were to be posed by the incumbent and the chair was to pass to the person presenting the best answers ; the incumbent could ill afford to publish all his discoveries ; that Roberval retained the chair for 41 years, until his death at the age of 73, is proof of the fertility of his mind ". The appointment of a professor of mathematics in the America of 1785 is also of interest. " In April 1785 John Kemp was appointed to teach mathematics for one year at Columbia College in New York City. At the end of that time there was a public examination of his class, in which each student was required to draw a number out of a box and demonstrate without further assistance the problem or theorem in Euclid to which it referred. The examination was unusually successful, and this convincing exhibition of Kemp's mastery of his subject and his ability to teach it led to his appointment as professor of mathematics and natural philosophy in 1786 at the age of twenty-three." The views of Samuel Johnson, the first President of King's College (at present Columbia University), deserve attention even to-day. He was sole tutor at Yale from 1716-1718, later pastor of a church at Stratford, Connecticut, and from this post was called to King's College. " As the understanding of Young Persons, for the first fifteen or sixteen years of their Life, are not ripe enough to enter into the sublime Studies of *Philosophy*, it is necessary that during this Stage they should be chiefly employed in the Study of *Philology*, or the Languages, to which should be added the first Things in *Mathematic*, both which are most level to their juvenile Capacities, as they chiefly depend on the Imagination and Memory, which in Youth are most vigorous and tenacious."

Readers who like science combined with a social background will enjoy reading the life and inventions of Samuel Morey, who was born in Hebron, Connecticut, in 1762. He shared the fate of most inventors and other persons ahead of their times in being continually misunderstood and hampered by irrelevant issues. " The trial trip was made on Sunday morning in 1793. The machinery and a small supply of wood for fuel left barely room enough in the boat for the inventor and his assistant, but despite its diminutive size the craft proved capable of moving at the rate of four miles per hour, even against the strong current in mid-stream. Fearing the ridicule of skeptical neighbours, Morey chose Sunday morning for the trial of his invention, relying upon the church-going propensities of the villagers to protect him from observation."

The book is essentially a browser's paradise. In it will be found things new and old, mathematical and human. W. P. M.

The Study of the History of Mathematics. By G. SARTON. Pp. vi, 113. $1.50. 1936. (Harvard University Press)

According to the sub-title this book is " the substance of the inaugural lecture of a course on the history of mathematics in Harvard University ", but even when this description is interpreted as the author intends it to be it is inadequate, for it applies to only one part of the book and there are three others, a note on the history of modern mathematics, a bibliography of the history of mathematics, and a list of more than a hundred modern mathematicians, with details of at least one biography or biographical notice of each of them and an exact statement as to issue of collected works and correspondence.

The note consists chiefly of encouragement and advice for the mathematician who has no wish to become a historian, but is sufficiently interested in the development of his own branch of the subject to contribute material for the historian to utilise.

The bibliography is selected, and is annotated, in part explicitly, and in part implicitly by references to *Isis*. Dr. Sarton speaks only of books with which he is himself acquainted, and his notes are the more valuable on that account. Occasionally one may question a judgment, as when the immense qualitative difference between Muir's history of the theory of determinants and Dickson's history of the theory of numbers is ignored, but one knows that an opinion is being expressed, not quoted. If Dr. Sarton has ever tried really to use the Royal Society's *Subject Index of Pure Mathematics*, his forbearance in refraining from caustic criticism is remarkable. The account of the later years of the *Revue Semestrielle* is somewhat misleading, for not only did this journal belie its title completely from 1932 onwards by appearing six times a year, but it has now ceased to exist in any form, and there is no prospect even of a collective index of the last five volumes.

Dr. Sarton observes of the historian of one branch of mathematics that he has often overlooked very important matters because these are included in papers specifically devoted to other subjects. He must blame his own subtitle if this book is slow to take its place as the valuable work of reference which it undoubtedly is. E. H. N.

Mathematics of Modern Engineering. I. By ROBERT E. DOHERTY and ERNEST G. KELLER. Pp. xxi, 314. 17s. 6d. 1936. (John Wiley, New York; Chapman & Hall)

Here, according to the title-page, we have "one of a Series written in the interest of the Advanced Course in Engineering of the General Electric Company". Its joint-authors are Dean of Engineering, Yale University, and Associate Professor of Applied Mathematics, University of Texas, respectively, who introduce themselves in their preface as "a mathematician who has worked in engineering and an engineer who has worked with mathematics". They claim the book to be "first, a guide in bridging the gap in engineering between physics and mathematics by the scientific method; and secondly, a presentation, suitable for engineers, of those aspects of mathematics which the experience of a large manufacturing organisation in dealing with electrical and mechanical problems has indicated to be of value to engineers".

The text begins with a six-page chapter headed "Mathematical Formulation of Engineering Problems", which clarifies the ideas underlying the translation of a physical law into a mathematical equation, and brings out the need for clear thinking and the use of definite criteria, particularly in the initial stages of the process.* The second chapter, which occupies more than one-half of the book, is justly named "Basic Engineering Mathematics", for it provides the bulk of that mathematical apparatus with which a scientific engineer should be provided. It contains six main sections, of which the first is an adequate treatment on classical lines of ordinary linear differential equations with constant coefficients, showing and illustrating by examples how such equations and systems of equations arise and are solved, and how the solutions are interpreted in practice. Then there is a section on determinants and their application to simultaneous algebraic and differential equations, followed by one on Fourier series as applied to the usual types of functions with a finite number of humps and jumps, here aptly termed "engineering functions", together a practical method of harmonic analysis suitable for the oscillogram. The next topic in the chapter is the numerical solution of algebraic and transcendental equations. Newton's method is

* Can it be that we recently maligned the *slug*, which we learn to be approximately the mass of 4 gallons of water? The engineer does need a distinctive and practical unit of mass, but its name might have been more happily chosen.

briefly discussed, but is left unsupported by any complementary method, so that the accuracy of the approximation remains somewhat vague. The authors seem to favour Graeffe's method, to which they devote 32 pages, referring in particular to the complex roots of algebraic equations. The fifth section is given to dimensional analysis, which, if used with discretion, frequently provided the engineer with a formula whose derivation on purely mathematical lines would be a matter of considerable labour. The last section in the chapter is on graphical and numerical methods of solving differential equations; it is most instructive and should be very useful, although it follows certain practices abhorrent to the trained computer.

The third chapter expounds the methods and symbolism of vector analysis, leading up to the derivation of partial differential equations of mathematical physics and to the beginning of vector magnetic theory, in which there is ample scope for further development. This, in brief, is the content of the first three sections of the chapter, which is excellent up to this point. The concluding section requires considerable elucidation on behalf of readers who are here to make their first acquaintance with dyadics. In the fourth and last chapter, the Heaviside operational calculus is based upon the Cauchy integral theorem. The greatest of the difficulties that this method presents are of a purely mathematical nature and need not trouble the engineer. Minor difficulties certainly remain, but they are well worth the effort of surmounting, an effort amply rewarded by the extraordinary power and scope of the operational methods, which seem, as a practical device, likely to supersede the older methods for obtaining the steady-state solution of the differential equations of a network. There is a useful bibliography, arranged to correspond with the sections of the text, a list of answers to the questions set as examples, and a comprehensive index.

It would be interesting to learn how this book fits in actual fact into the engineering curricula of American colleges. Admittedly it was prepared for a course " which comprises a selected group of engineering graduates just out of college ", but if it " should be useful as well in college in both undergraduate and graduate engineering work ", we can only conclude that the mathematical teaching in American engineering faculties is more far-reaching if not more thorough than that on this side of the Atlantic. Indeed, we fear that none but the very cream of our first-class graduates in engineering would really be able to appreciate the great value of what the book contains, and to master and use its contents. For though we have ventured to indicate one or two points where the book falls short of perfection, we believe that it would be of great value to a properly equipped reader, and as such should be found in the library of every institution that claims to teach scientific engineering.

E. L. I.

Differential Equations in Applied Chemistry. By F. L. Hitchcock and C. S. Robinson. Second edition, revised and enlarged. Pp. viii, 120. 7s. 6d. 1936. (John Wiley, New York; Chapman & Hall)

This book, which is intended for chemists and chemical engineers, gives the simple mathematical background of the typical equations which such students are called upon to set up and solve. The exposition is plain and vigorous, the examples eminently practical, and the working-out of those in the text sufficiently detailed to remove any suggestion of the occult which is sometimes apt to obscure the mathematical processes presented for the consumption of the plain chemist. Processes of the first and second order are dealt with; these are followed by simultaneous processes and equations of flow, a concluding chapter being devoted to successive approximations and numerical

solutions. The book is well made and printed, and of undoubted value to the chemical student; but, judged by English standards, the price is perhaps rather on the high side. F. M. B.

Einführung in die Theorie der Quadratischen Zahlkörper. By HEINRICH W. E. JUNG. Pp. viii, 150. RM. 5.80. 1936. (Jänecke, Leipzig)

In this book the author has given us in very compact form a fairly complete account of the "Quadratische Zahlkörper". This volume follows his *Einführung in die Zahlentheorie*, and the same notation is used. As this notation is rather unusual, and is not explained, the reader must refer to the earlier work, or indulge in guessing. The latter method is quite possible to anyone with any knowledge of the subject. The treatment of the subject is interesting and usually very clear. In the opinion of the reviewer, however, the fundamental principles of the subject are somewhat obscured by the method of introducing "ideals". For this drawback there is perhaps some compensation in the simplicity of the treatment.

The printing and general arrangement are clear and attractive. Also, in spite of a good deal of detailed calculation, I have found very few misprints. There is one on p. 25, but it is so obvious as to present no real obstacle to the reader. G. K. S.

Einführung in die Mechanik des Fluges. By W. MÜLLER. Pp. viii, 115. RM. 4.80. 1936. (Jänecke, Leipzig)

This little book, by one of the leading German writers on hydro- and aerodynamics, displays all the author's usual clearness of exposition and conciseness of expression, and is an excellent introduction to this interesting subject, ample for almost anyone but the specialist.

It contains four chapters. In the first, an account, surprisingly comprehensive considering the size of the book, of the facts of fluid motion, and of the aerodynamic forces acting upon the various parts of an aeroplane, is given. The second chapter is devoted to the various cases of steady motion, while accelerated motion—take-off, landing, looping, etc.—is treated in Chapter III. The last chapter contains a simple treatment of some of the more important considerations connected with the stability of an aeroplane in flight.

From the point of view of most of the readers of this review, the book can be regarded as a fairly rich storehouse of practical problems in mechanics, with a considerable range of mathematical difficulty, the topical interest of which cannot but enhance the interest, and consequently improve the performance, of any of our classes. A numerical example is given in nearly every important case. W. G. B.

Hydrostatics. By A. S. RAMSEY. Pp. viii, 169. 7s. 6d. 1936. (Cambridge University Press)

This textbook completes Mr. Ramsey's series of books on mechanics at this level, *i.e.* "for the use of first year students at the Universities, and for the higher divisions in schools". It covers the usual ground, from "pressure at a point" to metacentre, in a competent manner, and contains the usual generous sets of exercises at the ends of the chapters. There is also a chapter on Capillarity. In our opinion, it is more suited to the first-named group of students for whom it is written than the second.

One serious criticism can be levelled. It is surely time that the terms "mass-centre", "centre of gravity", and "moment of inertia" should be restricted to their rightful place, and not applied to such purely geometrical entities as surfaces and volunes. The author is not consistent, however, and

one notes with relief the use of the term " centroid " in the chapter on stability of equilibrium. W. G. B.

Spherical Astronomy. By W. M. SMART. Second Edition. Pp. xii, 430. 21s. 1936. (Cambridge University Press)

The publication of the first edition of Dr. Smart's work in 1931 filled what had become a serious gap among astronomical textbooks. The present edition differs partly in a number of corrections to the original text, and partly in the addition of three new appendices.

The corrections do not affect the pagination of pages 1 to 403, the body of the original book, and are mainly concerned with matters of detail. Perhaps the most important is the rectification of Section 137, which deals with the thorny subject of precession.

It is unfortunate that Section 134 has escaped revision, since it includes an erroneous explanation of the two nutational terms which have as arguments twice the solar and lunar longitudes respectively. For instance, the term $l \sin 2\odot$ in equation (12), as is evident from its independence of the longitude of perihelion, is not caused by the eccentricity of the earth's orbit ; it is really due to the fact that the solar disturbing couple is not constant, but vanishes when the sun is on the equator, that is, at the equinoxes. An entirely similar error affects the author's explanation of the corresponding lunar term in equation (13).

It may be pointed out as a minor matter that the value of 299,860 km./sec. used for the velocity of light *in vacuo* could usefully have been revised, in view of the values published by Michelson and his collaborators (299,796 in 1927 and 299,774 in 1935).

The first and most considerable of the new appendices deals in ten pages with the method of dependences. This relates to the photographic determination of the position of a selected object (comet, minor planet or parallax-star) in a field of reference-stars. The method is not without mathematical interest and now occupies an important position in astronomical practice, so that the appendix constitutes a very welcome addition.

The second appendix consists of two pages devoted to the subject of stellar magnitudes. The matter is of an elementary nature from an astronomical point of view, but its inclusion will be of assistance to mathematical students who are being given a dose of astronomy. The third appendix, four pages long, contains a useful account of the theory of the coelostat.

It is distinctly a matter for satisfaction that the demand has been sufficient to evoke this revised and enlarged edition, which will consolidate the position of Dr. Smart's book as the standard English work on its subject. A. F.

Examples in Finite Differences, Calculus, and Probability. By HARRY FREEMAN. Pp. 86. 8s. 6d. 1936. (Cambridge University Press)

Mr. Freeman's *Elementary Treatise on Actuarial Mathematics*, published five years ago, has now become standard in this country for study and reference. It contained a large number of exercises and examples. The present book supplements these by a further 400 varied examples, with very useful notes and hints for the solution of some eighty of them.

These exercises seem admirably chosen for their purpose. For one thing they are fresh and not shop-soiled ; even the old unavoidable chestnuts have been given a new polish. Random sampling among the questions on probability disclosed only one error, a typographical one, 2917 for 2197 in the answer to Ex. 293 (7). One may suggest, too, that in addition to the two approaches to Ex. 331 outlined in the hints, there is a third, simpler and more

natural than either, by which the problem may be generalised to the case of m black and n white balls, of which r white are to be drawn.

Example 102 refers to "the usual meaning" of $x^{(-m)}$. No doubt the meaning given on p. 39 of the author's other textbook is intended. If so, we must remark that factorials of positive and negative index are then inconsistent. The consistent definition of $x^{(-m)}$ is that given in Milne-Thomson's *Calculus of Finite Differences*, p. 44, namely, $[(x+1)(x+2)\ldots(x+m)]^{-1}$. In Examples 104 and 105 Lidstone's suggested notation for binomial coefficients might have been adopted with advantage.

It is hardly necessary to state that the format and printing are excellent.

A. C. A.

Les Conditions de Monogénéité. By D. MENCHOFF. Pp. 52. 15 fr. Actualités scientifiques et industrielles, 329; exposés sur la théorie des fonctions, III. (Hermann, Paris)

It is to be deplored that, before writing a book of such obvious interest to students of the theory of functions of a complex variable, the author does not take the trouble to define the sense in which he uses the word "holomorphic", as this does not happen to coincide with that generally employed by writers of current textbooks on the subject. Apparently in the sense in which M. Menchoff uses it, the function $f(z)$ is said to be holomorphic in a domain D if $\int f(z)dz$ taken round any simple closed rectifiable contour lying in D vanishes, no account whatever being paid to its differentiability in D.

Starting with this definition, the reader should have no difficulty in appreciating the subject-matter at his disposal, which contains, amongst other things, an account of the refinements that have been made to Cauchy's theorem during recent years (by the author himself and others) by relaxing the conditions relating to the differentiability of the function concerned, even to the extent of allowing for the existence of certain exceptional sets of points in which no such conditions are satisfied.

In addition, the book contains several chapters dealing with various types of conditions, under which a function is either holomorphic in a given domain, or differentiable (monogenic) at a given point; leaving the reader with the inevitable impression that this particular branch of the theory of functions is, as yet, comparatively unexplored.

J. H. P.

Plane Trigonometry. Pp. xii, 152. **Six-place Tables.** 5th edition. Pp. xxiii, 156. By E. S. ALLEN. Bound in one, 6s. 1936. (McGraw-Hill)

The author has set out to give a clear and precise account of the elements of the subject; emphasising plainly the points of difficulty that confront the student, at the same time offering many helpful suggestions for overcoming and even avoiding them, and has done it remarkably well.

The idea of defining trigonometrical functions of any angle, without reference to analytic geometry, seems, however, a little unnecessary (especially in view of the fact that the conception of Cartesian coordinates is so fundamental in all branches of mathematics), but academically its interest is not to be disputed.

Chapter 7, which gives an exhaustive account of the Solution of Triangles, is particularly well set out, and with the help of the excellent set of tables available at the end of the book the reader should be in a position to solve any triangle with the utmost ease.

A discussion of the elementary theory of complex numbers and their uses in trigonometrical and other mathematical calculations brings to a close a book which should prove invaluable to all those teaching and learning the subject.

J. H. P.

The Groundwork of School Geometry. By F. A. J. RIVETT. Pp. 128. 2s. 1936. (Arnold)

Lest the eye-catching title should mislead, it may be well to mention first that this attractive little book might equally well be called "Stage A Geometry" or "First-year Work in Geometry". As such there is a certain freshness in the treatment and attractiveness in the way in which it is set out, which should make it definitely a useful book. There are a good many points open to criticism or question, but if these are named it must not be supposed that on the balance they override the merits.

The preface speaks of "reference wherever possible to three-dimensional work both in discussion and in the making of models", but this is scarcely carried out. The usual pictures of solids appear in the first page with the addendum "Models of these and other solids will be made later; see p. 102", and the intermediate work is mainly plane geometry after two exercises (Ex. 1 and 2) of questions about the solids. The book therefore does not adopt the method in which the construction of nets of solids and their models and sketches of them is used to introduce the use of instruments, but the more usual method in which these are introduced in connection with plane geometry only.

The instruments are introduced one by one; ruler, pp. 13-16; compasses, pp. 17-22; set square for right angles, pp. 24-5; protractor, pp. 32-3; set square for parallels, pp. 52-3. The treatment is thus systematic. Some attractive problems are given on pp. 21-2 under the heading "Using your compasses". On p. 25 "our constructions are likely to be more accurate if they are performed with ruler and compasses alone"; the truth of this is more than doubtful, and it is, anyhow, not the real reason for using ruler and compasses instead of set squares.

On p. 52 an important matter is dealt with in a somewhat disingenuous way.

"Lines are parallel if they are drawn in the same direction, *i.e.* if they are in the same plane and make equal angles with any transversal. This works 'both ways', and we can say that: If two lines are parallel then the corresponding angles made by them with any transversal will be equal."

"This works both ways" is rather nice English for "the converse theorem is true", but the weak spot is that in the first case "any transversal" means any *one*, while in the second case it means "any and *every*". On p. 57 something is done to repair this, for Ex. 26 tests by a series of measurements the theorem that "if two straight lines make equal corresponding angles with any one transversal, they must make equal corresponding angles with any other transversal". This was assumed on p. 52, and it would have been honest to have said so.

The first time Q.E.D. is used it is explained thus:

"[The letters Q.E.D. are often used for 'Answer' to a problem in Geometry. The letters stand for *quod erat demonstrandum*, meaning 'which was to be proved'. You can put them in opposite the last line of the proof if you like.]" Would it not improve this note to omit the first and third sentences?

The angles of a polygon are introduced by "careful measurement" in degrees from 4 up to 8 sides, and after that the proof is given. The preliminary exercise seems too long and dull.

Under loci, various attractive cases are given at once—that of the mid-point of a rod with ends sliding on two perpendicular straight lines, an ellipse, a spiral, etc. Symmetry is introduced by means of an attractive "ink-demon". Congruence is introduced nicely and the discussion is followed by an ample set of examples; this is definitely an excellent section.

There is then a short discussion of area and the length of curved lines. It

is curious to see the usual figure for proving a parallelogram equal to a rectangle given merely for drawing on squared paper and counting of squares, especially after congruent triangles have been done so thoroughly. Then come " construction of models of solids " and " field work ". These could presumably be inserted at intervals throughout the course (though there seems to be no suggestion that this should be done). They do not seem in their place here. Finally come revision examples, and a list of " some things we have stated, discovered or proved ".

If there are things that offend in this book, there is certainly much to attract.

C. O. T.

Fundamental Geometry. Second Series. II. By P. B. BALLARD and E. R. HAMILTON. Limp cloth, 1s.; paper, 10d. 1936. (University of London Press)

In the first series Dr. Ballard and Mr. Hamilton promised a sequel, and this is now in course of publication. The present volume covers drawing to scale, similar figures, areas and volumes. Towards the end, when the angles of triangles and polygons are dealt with, the first traces of logical reasoning begin to creep in, as distinct from the knowledge of space with which the greater part of the book is concerned. The original idea of the first series was to introduce geometry to quite young children through the study of shape. In this second series this idea seems to have receded into the background and the treatment is more on the usual lines. But many of the good features of the earlier volumes have been retained—the examples, for instance, are well within the range of the child's experience—and, judging from this one volume, the series should prove a useful " Stage A " textbook.

J. H. H.

Revision Algebra for School Certificate. By A. C. HEATH and H. RUSSELL DAVIS. Pp. 71. 1s. 3d. 1936. (University of London Press)

This collection of rather more than 200 exercises of the harder types commonly met with in School Certificate and Matriculation examinations is arranged with explanatory paragraphs in twelve chapters, and is well designed for the use of those candidates who aim at the " credit " standard in such examinations.

In Chapter XI, which deals with indices and logarithms, perhaps it would be more natural to interpret $a^{p/q}$ as $(\sqrt[q]{a})^p$ rather than as $\sqrt[q]{(a^p)}$, just as the primary meaning of p/q is $\frac{1}{q} \times p$ rather than $1/q$ of p. In Chapter XII, which deals with progressions, perhaps it would be better to express S_n for an arithmetic progression as $n\left(\frac{a+l}{2}\right)$ rather than $\frac{1}{2}n(a+l)$, and thus emphasise the fact that the average of all is the same as the average of the first and last.

W. J. D.

Exercises in Algebra. I-III. By H. F. BROWNE. Pp. 160 each. 2s. each. 1936. (Ginn)

This is mainly a vast collection of carefully graduated exercises forming a three years' course of preparation for School Certificate examinations, each part corresponding to the work of one year, while Part III completes the course up to " Credit " standard and covers the syllabus of the " Advanced " paper. Each volume also contains a very useful set of revision papers.

Worked examples, often accompanied by " rules " and " definitions ", precede each set of exercises. These rules are sometimes proved, sometimes merely stated and illustrated, and then collected at the end of each volume.

At the end of Part III, 90 rules are collected together in 6 pages, and 104 definitions in the next 7 pages, these 13 pages forming a convenient index to the entire work. Some teachers would, perhaps, prefer to postpone many of these " rules " until their pupils had themselves discovered them, sometimes in a different form.

The work contains many excellent features; for instance, the checking of equations and problems after solution, the changing of the subject of a formula, the making of expressions, the treatment of indices and common logarithms.

In dealing with " graphs " the *axes* are defined as " the straight lines which show the values of the variables ". Is it not often more convenient to mark the *scales* not along the *axes* but along lines which do not meet at the *origin* ?

In Part III, p. 63, the rule for " changing the base of a logarithm " is proved by introducing two extra symbols. Yet on the next page the equation $2^x=7$ (which means " Find $\log_2 7$ ") is worked much more directly. If logarithms to base a are known and the logarithm of any number N to a new base b is required, we have merely to solve the equation $b^x=N$ with the aid of a table of logarithms to base a.

In the same Part, p. 99, the proof that when $-1<r<1$, then $r^n\to 0$ when $n\to\infty$, is not convincing.

The whole work is beautifully printed and well arranged, and the very numerous exercises are full of interest. There are no answers; but, perhaps, these are, or will be, available for teachers. W. J. D.

Das Grenzgebiet der elementaren und höheren Mathematik. By K. Kommerell. Pp. viii, 249. 18s. 6d. 1936. (Koehler, Leipzig)

Most of us must have been tempted at times to write a book of mathematical essays—not a connected, encyclopaedic treatise, but a set of chapters all more or less in the same line of business, presenting facts or aspects of facts which we have found interesting and for which there is no room in the bread-and-butter treatises which we use every day. The boundary of school and university mathematics is fertile ground, through which, however, the usual path is narrow. Professor Kommerell's book gives us an opportunity of exploration in this domain. It might well be read by a mathematical scholar just going up to the university, and it is emphatically a book for those teachers who believe that their knowledge of a subject should not be confined to those parts of it which they intend to present to their pupils.

The first of the three main sections deals with limit processes. There is the Newton iteration for the square root, and a very neat arrangement for calculating roots and logarithms by radix fractions in the binary scale. Then comes the calculation of π, being an account of methods due to Nicholas of Cusa and Huygens. The square root algorithm links up very naturally with Gauss' arithmetico-geometric mean and the calculation of elliptic integrals. Finally in this section there are some chapters on Diophantine equations and continued fractions, which I found most attractive because of the geometrical development and illustrations.

It is a little surprising to find that the second main section (Geometrical Transformations) opens with an elementary, though very systematic, account of inversion, a study almost invariably included in English school courses, but we soon pass to self-inverse (anallagmatic) curves and thence to an elementary and delightful account of Dupin's cyclides. Conical projection follows, viewed both geometrically and analytically, and general collineations, with applications to the Lorentz transformation, to deformable quadrics and other topics.

Finally, vectors, an elementary acquaintance with the concept and its

fundamental algebra being assumed, are related to various ideas in spherical trigonometry and line coordinates. The last two chapters in this final section, one on cubic and biquadratic equations preliminary to the other on the construction of the regular polygon of 17 sides, seem unrelated to the rest of the section, but in themselves are typical both of the general scope of the work and the way in which it has been carried out.

The book is neatly bound, with very clear print and diagrams, and is full of interesting material not always easily found elsewhere, lucidly and attractively expounded. T. A. A. B.

Mathematische Formelsammlung. By O. Th. Bürklen and F. Ringleb. 3rd edition. Pp. vii, 270. RM. 1.62. Sammlung Göschen, 51. 1936. (Walter de Gruyter, Berlin)

I cannot refrain from describing this as a pocket omnibus; it caters for all elementary mathematics (and some applications), and it can be carried in the pocket. The sections are: Arithmetic, Algebra, Theory of Numbers, Series, Plane Geometry, Stereometry, Plane Trigonometry, Spherical Trigonometry, Mathematical Geography and Astronomy, Analytical Geometry of two dimensions, Analytical Geometry of space and Vector Calculus, Differential Calculus, Integral Calculus, Differential Geometry, Differential Equations.

The contents are wider than might be imagined from the headings; Arithmetic includes formulae for determinants and for probability, Integral Calculus includes something about the complex variable and function theory, Differential Geometry includes the elementary formulae for twisted curves and surfaces.

In spite of the small page and the close packing, reading is easy, and I have detected no misprints. Two minor points: the formula for $\int dx/x$ is careless; the authors make the radius of curvature, rather than curvature, fundamental, and their remarks about its sign do not appear to be quite consistent.

The Sammlung Göschen always offers good value for money; this volume is above the average in this respect. T. A. A. B.

Classification for Works on Pure and Applied Science in the Science Museum Library. 3rd edition. Pp. 132. 5s. Postage extra. 1936. (Science Museum, South Kensington; H.M. Stationery Office)

Bad classification will diminish enormously the value of a library; and classification is a most difficult problem. Having laid down a clear scheme, where does Weyl's *Gruppentheorie und Quantenmechanik* go? Is Klein's *Ikosaeder* to appear in the section on geometry? Such questions continually arise. The present South Kensington scheme is, of course, decimalised, and has the approval of the International Institute of Documentation. As far as the mathematical classification goes, it appears to be clear and fairly logical, but hardly complete; there is a late-Victorian flavour about the entries. Integral functions and Eulerian integrals share an entry; Fourier series and almost periodic functions are not mentioned; integral equations and functional equations are entered twice, but the theory of functionals not at all.

As this list of peculiarities could be considerably extended, it seems to be fair to say that while the general scheme should prove useful, the particulars should be brought up to date. T. A. A. B.

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